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Approximate Solution of Quintic and Higher-Degree Equations

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Abstract

The general exact solution by radicals of the De Moivre equations of any degree is presented. But, since it is impossible to obtain the general exact solution of any quintic by radicals, except those of De Moivre or binomial, approximate method for solving any monic quintic, sextic, ... or 30th degree equation is proposed, namely, by decomposing them into two factors of degrees n–1 and 1, we obtain n unknown coefficients and n equations, which allow us to know all their roots, with a high degree of approximation.

Keywords: *Exact solution of De Moivre's equation by radicals of 3rd, 4th or any degree, Solution of any monic Quintic, Sextic, Septic, ... , or 30th degree equation by a general approximate method*

1. Introduction

So far there have been many unsuccessful attempts to solve the general quintic equation by radicals by mathematicians of all times: Green R F and Ehrenfried Walter von Tschirnhaus (1683); De Moivre (1706); Chen *et al.* (2017); Euler (1738); Jerrard (1859); Lavallee *et al.* (2005); Dummit (1991); and many others. However, since Paolo Ruffini (Dickson, 1926; King, 1996; Dummit, 1991), Niels Henrik Abel (Dickson, 1926; King, 1996; Dummit, 1991) and Évariste Galois (Dickson, 1926; King, 1996; Dummit, 1991) proved that the general quintic is not solvable by radicals, these attempts, have been greatly reduced. The meaning of the phrase by radicals refers to the roots of such equations as functions of the coefficients, obtained them by means of a finite number of algebraic operations. The main objective of this paper is to present an approach to solve any quintics or higher degree equations by an approximate method.

In this work we develop the foundations for obtaining the general formulae for solving the De Moivre Equations in Section 2.2. Since it was not possible to solve the general quintic by means of radicals, we instead developed an approximate method to solve equations of orders of degree equal to and greater than 5. The approximate method used to solve the general equation of degree n is described in Section 2.3. The exact method for reducing the equations of degrees 3 and 4 to De Moivre form and solving them by radicals is shown in Sections 3.1 and 3.2. The quintic, sextic and septic equations are solved, as mentioned above, by an approximation method in Sections 3.3, 3.4 and 3.5. Comments and conclusions are presented in Section 4.

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2. Foundations and Method

2.1. Binomial Equation

As is known any general equation of the n^{th} degree, without the n–1 internal terms, $x^n - X_0 = 0$, can be easily solved by radicals by the formula:

$$x_m = \sqrt[n]{X_0}e^{j\left(m\frac{2\pi}{n}\right)}, \text{ for } m = 0, 1, .., n\text{-}1 \quad ...(1a)$$

2.2. De Moivre equation

A general quintic monic equation $z^5 + \sum_1^5 Z_{5-j}z^{5-j} = 0$, can be reduced by sequential Tschirnhaus transformations of the 1st, 2nd and 4th degree, to the Bring-Jerrard normal form, $x^5 + qx + r = 0$. For achieving this, E. S. Bring, in 1786 used a 4th degree transformation whose coefficients were: ***c = d + γ, b = σd + ζ*** and ***a* = (*3pd* + *4q*)/5** (Green, 1683; Alexander *et al.*, 1786; and Adamchik and Jeffrey, 2003), in order to avoid degrees of the auxiliary equations greater than 4.

On the other hand, if it were possible to eliminate inter-sequential terms from a general monic equation, $z^n + \sum_1^n Z_{n-j}z^{n-j} = 0$, it results:

For odd n: $y^n + \sum_{p=1}^{\frac{n-1}{2}} Y_{n-2p}y^{n-2p} + Y_0 = 0$; ...(1b)

For even n: $y^n + \sum_{p=1}^{n/2} Y_{n-2p}y^{n-2p} = 0$ ...(1c)

and, if the coefficients can respond as: $Y_{n-2p} = (-1)^p C_{n-2p}\left(\frac{Y_{n-2}}{C_{n-2}}\right)^p = (-1)^p C_{n-2p}\alpha^p$, for $\alpha = \frac{Y_{n-2}}{C_{n-2}}$, the De Moivre equation of odd or even degree, Equations (2) and (3) below, arises:

$$\begin{cases} \text{For } n \text{ odd:} \quad y^n + \sum_{p=1}^{\frac{n-1}{2}}(-1)^p C_{n-2p}\alpha^p y^{n-2p} - C_0 = 0 \\ \text{For } n \text{ even:} \quad y^n + \sum_{p=1}^{\frac{n}{2}-1}(-1)^p C_{n-2p}\alpha^p y^{n-2p} - C_0 = 0 \end{cases}, \text{ or}$$

$$y^n - C_{n-2}\alpha y^{n-2} + \cdots + C_{n-2p}\alpha^p y^{n-2p} - \cdots \pm C_{n-2\left(\frac{n-1}{2}\right)}\alpha^{\frac{n-1}{2}} y - C_0 = 0 \quad ...(2)$$

$$y^n - C_{n-2}\alpha y^{n-2} + \cdots + C_{n-2p}\alpha^p y^{n-2p} - \cdots \pm C_{n-2\left(\frac{n-2}{2}\right)}\alpha^{\frac{n-2}{2}} y^2 - C_0 = 0 \quad ...(3)$$

As will be shown next, any De Moivre's equation can be solved by radicals.

In general, a monic equation of degree n, $z^n + \sum_1^n Z_{n-j}z^{n-j} = 0$, reduced to the De Moivre form Equations (2) and (3) can be compared with the expansion of $(u+v)^n$, converted into a De Moivre equation. Such expansion can be represented either by the well-known power of a binomial, or by an identity which has the natural De Moivre structure, i.e.:

$$(u+v)^n \equiv \binom{n}{0}u^n + \binom{n}{1}vu^{n-1} + \cdots + \binom{n}{n-1}v^{n-1}u + \binom{n}{n}v^n$$

$$(u+v)^n \equiv C_{n-2}(uv)(u+v)^{n-2} - C_{n-4}(uv)^2(u+v)^{n-4} + \cdots + C_0$$

The last identity is constructed by using the sum of symmetric terms within the first identity and completing the factors $(u+v)^i$ with the missing elements. Then, passing all terms to the left-hand side of the equal sign, we convert the identity into an equation. For example:

$$(p+q)^3 = p^3 + 3p^2q + 3pq^2 + q^3 = 3pq(p+q) + p^3 + q^3 \ \rightarrow \ \mathbf{(p+q)^3 - 3pq(p+q) - (p^3+q^3) = 0}$$

$$(p+q)^4 = p^4 + 4p^3q + 6p^2q^2 + 4pq^3 + q^4$$
$$= 4pq(p^2+q^2) + 6p^2q^2 + p^4 + q^4 + [4pq(2pq) - 4pq(2pq)]$$

$$= 4pq(p+q)^2 \ + p^4 + q^4 - 2p^2q^2 \ \rightarrow \ \mathbf{(p+q)^4 - 4pq(p+q)^2 - (p^2 - q^2)^2 = 0}$$

$$(p+q)^5 = p^5 + 5p^4q \ + 10p^3q^2 + 10p^2q^3 + 5pq^4 + q^5$$
$$= 5pq(p^3+q^3) + 10p^2q^2(p+q) + p^5 + q^5$$
$$+ [5pq(3p^2q + 3pq^2) \ - 5pq(3p^2q + 3pq^2)]$$

$$= \ 5pq(p+q)^3 - 5p^2q^2(p+q) + p^5 + q^5 \ \rightarrow \ \mathbf{(p+q)^5 - \ 5pq(p+q)^3 + 5p^2q^2(p+q) - (p^5+q^5) = 0}$$

$$(p+q)^6 = p^6 + 6p^5q \ + 15p^4q^2 + 20p^3q^3 + 15p^2q^4 + 6pq^5 + q^6$$
$$= 6pq(p^4+q^4) + 15p^2q^2(p^2+q^2) + 20p^3q^3 + p^6 + q^6$$

$$+ [6pq(4p^3q + 6p^2q^2 + 4pq^3) - 6pq(4p^3q + 6p^2q^2 + 4pq^3)]$$
$$+ [15p^2q^2(2uv) - 15p^2q^2(2uv)] \ + [2p^3q^3 - 2p^3q^3]$$

$$\rightarrow \ \mathbf{(p+q)^6 - 6pq(p+q)^4 + 9p^2q^2(p+q)^2 - (p^3+q^3)^2 = 0}$$

This feature of the power of a binomial, $(u+v)^n$, of naturally becoming the De Moivre structure, see Equation (4) below, allows us to take it as a model with which to compare Equations (2) and (3), and solve them using $y = u + v$, $\alpha = uv$, $C_{n-2} = n$ and $C_0 = \beta_n$.

$$(u+v)^n - n(uv)(u+v)^{n-2} + \cdots \mp C_{n-2p}(uv)^p(u+v)^{n-2p} \pm \cdots - \beta_n = 0 \qquad ...(4)$$

Moreover, and even better, the coefficients C_{n-2p} of Equation (4), n even or odd, can be structured as a right triangle, with n rows and p columns ($p = 0, 1, ...$), see Table 1, where the one-dimensional coefficients C_{n-2p} in Equation (4), when placed in the right triangle, change to a two-dimensional expression $C_{n,p}$. In this way, coefficients C_{n-2p} can be calculated as diagonal sums in a style similar to that of Pascal's or Tartaglia's triangle. The factors α^p in the terms of Equation (4), and the β_n, are also easily obtained. The following properties of the terms can be visualized as:

- For odd $n \geq 5$, absolute values of extreme internal coefficients in Equation (2) are

$$|C_{n-2}| = \left|C_{n-2\frac{(n-1)}{2}}\right| = n. \qquad \text{...(5a)}$$

The same definition, expressed in two dimensions, in Table 1 below, becomes:

$$|C_{n,1}| = \left|C_{n,\frac{(n-1)}{2}}\right| = n. \qquad \text{...(5b)}$$

- Filling the far right inside the even *n* lines with the value 2, or -2, helps us to build all the results of sums down. But, such numbers are not real coefficients! They are just filler numbers, to construct extreme diagonal sums (Equation 6 below) in Table 1.

Table 1: Rectangular Triangle: The Result of Diagonal Sums of the Coefficients is Displayed Vertically Downwards. This Rule Allows us Calculate all the Coefficients of the De Moivre Equation

$n\downarrow$ $p \rightarrow$	0	1	2	3	4	5	6	…	βn
1	1								$-(u+v)$
2	1	-2							$-(u^{2/2}+v^{2/2})^2$
3	1	-3							$-(u^3+v^3)$
4	1	-4	2						$-(u^{4/2}-v^{4/2})^2$
5	1	-5	5						$-(v^5+v^5)$
6	1	-6	9	-2					$-(u^{6/2}+v^{6/2})^2$
7	1	-7	14	-7					$-(u^7+v^7)$
8	1	-8	20	-16	2				$-(u^{8/2}-v^{8/2})^2$
9	1	-9	27	-30	9				$-(u^9+v^9)$
10	1	-10	35	-50	25	-2			$-(u^{10/2}+v^{10/2})^2$
11	1	-11	44	-77	55	-11			$-(u^{11}+v^{11})$
12	1	-12	54	-112	105	-36	2		$-(u^{12/2}-v^{12/2})^2$
13	1	-13	65	-156	182	-91	13		$-(u^{13}+v^{13})$

- For even $n \geq 4$, the last term occurs for $p = n/2$ and for $n \geq 5$ for $p = (n-1)/2$.

- Results of diagonal sums (including the filler numbers 2 or -2) respond to:

$$|C_{n,p}| = (-1)^p\left(|C_{(n-2),(p-1)}| + |C_{(n-1),p}|\right) \qquad \text{...(6)}$$

- If n is odd, $\beta_n = u^n + v^n$. But, if is even, $\beta_n = (u^{n/2} \pm v^{n/2})^2$, and sign $\pm$ inside β_n is given by, $\pm = (-1)^{(n/2-1)}$. Thus, if we consider *n* even and *n*/2 odd, sign within β_n is +; but, if *n*/2 is even, sign is –. Check this in Table 1:‘

2.3. Solving the General De Moivre Equation by Radicals

Next is obtained the general solution, by radicals, of the De Moivre equation, where, as will be seen, it depends only on α or Y_{n-2}, and β_n or Y_0. To do so, let us first compare the equation of degree *n* odd Equation (7), in De Moivre's form, with Equation (8), below

$$y^n - Y_{n-2}y^{n-2} + Y_{n-4}y^{n-4} - Y_{n-6}y^{n-6} + \cdots \pm Y_1 y - Y_0 = 0$$

$$y^n - C_{n-2}\alpha y^{n-2} + C_{n-4}\alpha^2 y^{n-4} - C_{n-6}\alpha^3 y^{n-6} \mp \cdots \pm C_{n-2\left(\frac{n-1}{2}\right)}\alpha^{\frac{n-1}{2}} y - C_0 = 0 \qquad \text{...(7)}$$

$$(u+v)^n - n(uv)(u+v)^{n-2} + \cdots \mp C_{n-2p}(uv)^p(u+v)^{n-2p} \pm \ \ldots \pm C_{n-2\left(\frac{n-1}{2}\right)}\alpha^{\frac{n-1}{2}}y - \beta_n = 0 \quad \text{...(8)}$$

where, for $y = u + v$, $Y_{n-2} = C_{n-2}\alpha = n(uv)$ and $Y_1 = C_1\alpha^{\frac{n-1}{2}} = n\alpha^{\frac{n-1}{2}}$:

$$\begin{array}{ll} Y_{n-2} = n\alpha & u = \frac{Y_{n-2}}{nv}, \\ \alpha = \frac{Y_{n-2}}{n} = uv\,; & v = \frac{Y_{n-2}}{nu}\,; \end{array} \quad \begin{array}{c} Y_{n-2p} = c_{n-2p}\alpha^p = c_{n-2p}\left(\frac{Y_{n-2}}{n}\right)^p \\ Y_0 = C_0 = \beta_n = u^n + v^n \end{array}$$

These relationships give rise to auxiliary equations and their solutions:

$$u^{2n} - Y_0u^n + \left(\frac{Y_{n-2}}{n}\right)^n = 0 \rightarrow u^n = \frac{Y_0}{2} + \frac{1}{2}\sqrt{Y_0^{\,2} - 4\left(\frac{Y_{n-2}}{n}\right)^n} \rightarrow u = \sqrt[n]{\frac{Y_0}{2} + \frac{1}{2}\sqrt{Y_0^{\,2} - 4\left(\frac{Y_{n-2}}{n}\right)^n}}$$

$$v^n = Y_0 - u^n \ \rightarrow v^n = \frac{Y_0}{2} - \frac{1}{2}\sqrt{Y_0^{\,2} - 4\left(\frac{Y_{n-2}}{n}\right)^n} \ \rightarrow \ v = \sqrt[n]{\frac{Y_0}{2} - \frac{1}{2}\sqrt{Y_0^{\,2} - 4\left(\frac{Y_{n-2}}{n}\right)^n}}$$

A basic solution immediately appears, $y_1 = u + v$. The solution for odd *n* arises by multiplying u and v by the unit roots, $\omega_m = e^{j\left(m\frac{2\pi}{n}\right)}$, giving exactly the necessary *n* roots, by using $y_k = \omega_{k-1}u + \omega_{n-(k-1)}v$, as follows:

$$y_k = \omega_{k-1}\sqrt[n]{\frac{Y_0}{2} + \frac{1}{2}\sqrt{Y_0^{\,2} - 4\left(\frac{Y_{n-2}}{n}\right)^n}} + \omega_{n-(k-1)}\sqrt[n]{\frac{Y_0}{2} - \frac{1}{2}\sqrt{Y_0^{\,2} - 4\left(\frac{Y_{n-2}}{n}\right)^n}} \quad \text{...(9)}$$

For:
$$\begin{array}{ll} y_1 = u + v = \omega_0u + \omega_nv & \\ y_2 = \omega_1u + \omega_{n-1}v & \\ y_3 = \omega_2u + \omega_{n-2}v & k = 1 \ldots n \\ :::::: & m = 0,1,\ldots,n \\ y_{n-1} = \omega_{n-2}u + \omega_2v & \omega_0 = \omega_n = 1 \\ y_n = \omega_{n-1}u + \omega_1v & \end{array} \quad \text{...(10)}$$

Now, comparing the equation of even degree *n*, in Equation (1) or Equation (11), and the De Moivre's form Equation (3), with the binomial form Equation (12), below:

$$y^n - Y_{n-2}y^{n-2} + Y_{n-4}y^{n-4} - \cdots \pm Y_{n-2p}y^{n-2p} \mp \cdots \pm Y_2y^2 - Y_0 = 0 \quad \text{...(11)}$$

$$y^n - C_{n-2}\alpha y^{n-2} + \cdots \pm C_{n-2p}\alpha^p y^{n-2p} - \cdots \pm C_{n-2\left(\frac{n-2}{2}\right)}\alpha^{\frac{(n-2)}{2}}y^2 - C_0 = 0 \quad \text{...(3)}$$

$$(u+v)^n - n(uv)(u+v)^{n-2} + \cdots \pm C_{n-2\frac{n-2}{2}}(uv)^{\frac{n-2}{2}}(u+v)^{n-2\frac{n-2}{2}} - \beta_n = 0 \quad \text{...(12)}$$

where, for $v = \frac{Y_{n-2}}{nu}$, $u = \frac{Y_{n-2}}{nv}$, and $Y_0 = C_0 = \beta_n = \left(u^{n/2} \pm v^{n/2}\right)^2$, two auxiliary equations, become created:

$$C_{n-2} = n;\ Y_{n-2} = n\alpha; \quad \alpha = uv;\ Y_{n-2p} = C_{n-2p}\alpha^p; \quad Y_0 = \beta_n = \left(u^{n/2} \pm v^{n/2}\right)^2$$

$$u^n - \sqrt{Y_0}u^{\frac{n}{2}} \mp \left(\frac{Y_{n-2}}{n}\right)^{\frac{n}{2}} = 0 \rightarrow u = \sqrt[\frac{n}{2}]{\frac{\sqrt{Y_0}}{2} + \frac{1}{2}\sqrt{Y_0 \pm 4\left(\frac{Y_{n-2}}{n}\right)^{\frac{n}{2}}}} \quad \text{...(13)}$$

$$v^{\frac{n}{2}} = Y_0 - u^{\frac{n}{2}} \rightarrow v^{\frac{n}{2}} = \frac{\sqrt{Y_0}}{2} - \frac{1}{2}\sqrt{Y_0 \pm 4\left(\frac{Y_{n-2}}{n}\right)^{\frac{n}{2}}} \rightarrow v = \sqrt[\frac{n}{2}]{\frac{\sqrt{Y_0}}{2} - \frac{1}{2}\sqrt{Y_0 \pm 4\left(\frac{Y_{n-2}}{n}\right)^{\frac{n}{2}}}} \quad ...(14)$$

The basic solution, $y_1 = u + v = \omega_0 u + \omega_n v$, and the others (with the same definitions), $y_k = \omega_{k-1} u + \omega_{n-(k-1)} v$, for even n, are:

$$y_k = \omega_{k-1}\sqrt[\frac{n}{2}]{\frac{\sqrt{Y_0}}{2} + \frac{1}{2}\sqrt{Y_0 \pm 4\left(\frac{Y_{n-2}}{n}\right)^{\frac{n}{2}}}} + \omega_{n-(k-1)}\sqrt[\frac{n}{2}]{\frac{\sqrt{Y_0}}{2} - \frac{1}{2}\sqrt{Y_0 \pm 4\left(\frac{Y_{n-2}}{n}\right)^{\frac{n}{2}}}} \quad ...(15)$$

$$y_1 = u + v = \omega_0 u + \omega_n v$$
$$y_2 = \omega_1 u + \omega_{n-1} v$$
$$y_3 = \omega_2 u + \omega_{n-2} v$$
$$::::::::: $$
$$y_{n-1} = \omega_{n-2} u + \omega_2 v$$
$$y_n = \omega_{n-1} u + \omega_1 v$$

$k = 1 \ldots n$

for $\omega_m = e^{j\left(m\frac{2\pi}{n}\right)}$, and the sign $\pm$ of $4\left(\frac{Y_{n-2}}{n}\right)^{\frac{n}{2}}$ is $\pm = (-1)^{\left(\frac{n}{2}-1\right)}$

$m = 0,1, \ldots, n$

The above proof indicates that any quintic or higher degree with the De Moivre structure (DMQ) can be solved by radicals.

2.4. Approximate Method for Solving any Monic Quintic, Sextic, … or 30th Degree Equation

Since, in general, it is not possible to reduce unsolvable quintic equations to the Binomial or De Moivre structure by equations of order less than five, as established by the theorems of Ruffini, Abel and Galois (Jacqueline, 2010), we will use the capabilities of the Sagemath software to obtain numerical approximations of the solutions of a system of equations, applied to any equation of degree equal to or greater than five.

To do this we expand the monic polynomial, $f(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_0$, as the product of two factors: $f(x) = (x^{n-1} + b_{n-2}x^{n-2} + b_{n-3}x^{n-3} + \cdots + b_0)(x - b) = 0$, and then set up and solve the system of n equations (representing the equality of their coefficients) with n unknowns (b_{n-2}, b_{n-3}, ..., b_0 and b). The solutions of such equations system are n groups of coefficients, or the total combinations, $C_{n,1} = n$.

Since any equation can be reduced to a monic equation and also it can be reduced to its normal form, without the first three internal terms (Chen *et al.*, 2017), $y^n + a_{n-4}x^{n-4} + \cdots + a_0$, or Bring-Jerrard normal form (BJ), we will always start from this structure for ease of solving (however, the approximate method is also valid for any monic equation in its complete structure). In the following sections we will present the approximate method we have developed to solve this type of monic equations of order five or higher.

3. Some Exact Results

3.1. Solving the Cubic Equation by Radicals

A first degree Tschirnhaus transformation, $y = x + \frac{b}{3}$, is applied to the general monic equation of the 3rd degree to eliminate the 2nd term. This transforms the resultant directly into the De Moivre cubic structure. A flowchart summarizing this calculation is:

$$f(x) = x^3 + bx^2 + cx + d = 0 \downarrow \qquad \leftarrow \qquad x = -\frac{b}{3} + y$$

$$\overset{\downarrow}{\rightarrow} \ \& \quad g(x) = y - \left(x + \frac{b}{3}\right) = 0 \rightarrow Res[f(x), g(x)] = y^3 - Y_1 y - Y_0 = 0 \rightarrow y_{1,2,3} \uparrow$$

Figure 1. Flowchart showing the steps to solve the Cubic Equation

The expressions of Y_1 and Y_0, calculated through the Sylvester matrix become:

$$\left\{\begin{array}{c} Y_1 = -\left(c - \frac{1}{3}b^2\right) \\ Y_0 = -\left(\frac{2}{27}b^3 - \frac{1}{3}bc + d\right) \end{array}\right\} \quad ...(16)$$

$$x_k = -\frac{b}{3} + \omega_{k-1}\sqrt[3]{\frac{Y_0 + \sqrt{Y_0{}^2 - \left(\frac{Y_1}{3}\right)^3}}{2}} + \omega_{n-(k-1)}\sqrt[3]{\frac{Y_0 - \sqrt{Y_0{}^2 - 4\left(\frac{Y_1}{3}\right)^3}}{2}} \qquad \begin{array}{l} x_1 = -\frac{b}{3} + \omega_0 u + \omega_3 v \\ x_2 = -\frac{b}{3} + \omega_1 u + \omega_2 v \\ x_3 = -\frac{b}{3} + \omega_2 u + \omega_1 v \end{array} \quad ...(17)$$

As seen, by using the solution for y_k, odd n in (10), and undoing the change to $x = -\frac{b}{3} + y$, we obtain a formula similar to Cardano's [5].

3.2. Solving the Quartic Equation by Radicals

Starting from a reduced quartic without the 2nd term (to simplify, not required), $f(x) = x^4 + cx^2 + dx + e = 0$, we apply a 2nd degree Tschirnhaus transformation, $g(x) = x^2 + px + q - y$, in order to eliminate the 2nd and 4th terms from the resultant and obtain its De Moivre form (DMF), whose general solution is directly given by equation (13):

$$f(x) = x^4 + cx^2 + dx + e = 0 \downarrow \qquad \leftarrow \qquad x_{1,2} = \frac{-p \pm \sqrt{p^2 - 4(q - y_k)}}{2}$$

$$\stackrel{\downarrow}{\rightarrow} \& g(x) = x^2 + px + q - y = 0 \xrightarrow{Res.} \sum_{j=0}^{4} Y_j y^j = 0 \xrightarrow[Y_1=0]{Y_3=0} y^4 + Y_2 y^2 + Y_0 = 0 \xrightarrow{(15)} y_k \uparrow$$

Figure 2: Flowchart of the steps to solve the Quartic Equation

Observe that undoing the 2nd degree transformation $g(x)$ means that each value of the resultant y, gives rise to two values in x. See some details of this procedure:

The resultant of $f(x)$ and $g(x)$, is: $f(y) = y^4 + Y_3 y^3 - Y_2 y^2 + Y_1 y - Y_0 = 0$ ...(18)

$$\text{where} \left\{\begin{array}{c} Y_3 = c - 2q \\ Y_2 = -(cp^2 + c^2 + 3dp - 6cq + 6q^2 + 2e) \\ Y_1 = dp^3 - 2cp^2 q + cdp + 4ep^2 - 2c^2 q - 6dpq + 6cq^2 - 4q^3 - d^2 + 2ce - 4eq \\ Y_0 = -(ep^4 - dp^3 q + cp^2 q^2 + cep^2 - cdpq - 6dpq - 6ep^2 q + \cdots. \\ \ldots + c^2 q^2 + 3dpq^2 - 2cq^3 + q^4 - dep + d^2 q - 2ceq + 2eq^2 + e^2) \end{array}\right\} \quad ...(19)$$

Making $Y_3 = Y_1 = 0$ we obtain, the value $q = c/2$ and three values of p coming from a 3rd degree equation. With Y_2 and Y_0 known we have: $y^4 - Y_2 y^2 - Y_0 = 0$, whose roots, written directly from the general solution (15) for even DMF, are:

$$y_k = \omega_{k-1} u + \omega_{n-(k-1)} v = \omega_{k-1}\sqrt{\frac{\sqrt{Y_0} \pm \sqrt{Y_0 + Y_2{}^2/4}}{2}} + \omega_{n-(k-1)}\sqrt{\frac{\sqrt{Y_0} \pm \sqrt{Y_0 - Y_2{}^2/4}}{2}} \qquad \begin{array}{l} y_1 = \omega_0 u + \omega_4 v \\ y_2 = \omega_1 u + \omega_3 v \\ y_3 = \omega_2 u + \omega_2 v \\ y_4 = \omega_3 u + \omega_1 v \end{array} \quad ...(20)$$

The roots x_i of $g(x) = x^2 + px + q - y = 0$, become:

$$x_k = \frac{-p \pm \sqrt{p^2 - 4(q - y_k)}}{2} = \frac{-p \pm \sqrt{p^2 - 4(q - (\omega_{k-1}u + \omega_{n-(k-1)}v))}}{2} \quad ...(21)$$

$$x_k = \frac{-p \pm \sqrt{p^2 - 4\left[q - \left(\omega_{k-1}\sqrt{\frac{\sqrt{Y_0} \pm \sqrt{Y_0 + Y_2{}^2/4}}{2}} + \omega_{n-(k-1)}\sqrt{\frac{\sqrt{Y_0} \pm \sqrt{Y_0 - Y_2{}^2/4}}{2}}\right)\right]}}{2} \quad ...(22)$$

This would be a Cardano-style version of the solution of the quartic equation.

However, notice that the De Moivre quartic equation, $y^4 - Y_2 y^2 - Y_0 = 0$, would also admit to be solved in a shorter way making the change $y^2 = z \rightarrow y = \pm\sqrt{z}$, reducing it to a quadratic equation, $z^2 - Y_2 z - Y_0 = 0$, which would greatly simplify its solution. See this formula not depending on *u* and *v:*

$$y = \sqrt{z} = \pm\sqrt{\frac{Y_2 \pm \sqrt{Y_2{}^2 + 4Y_0}}{2}} \rightarrow x = \frac{-p \pm \sqrt{p^2 - 4(q - y)}}{2} = \frac{-p \pm \sqrt{p^2 - 4\left[q \mp \sqrt{\frac{Y_2 \pm \sqrt{Y_2{}^2 + 4Y_0}}{2}}\right]}}{2} \quad ...(23)$$

In the first case, or Cardano's case, six sets of possibilities arise for the four roots of the quartic equation from the three values of *p*, and in the quadratic three sets arise, but only one is true: the one that returns the original equation, $x^4 + cx^2 + dx + e = 0$; i.e., we have to check all the possible solutions to see which one is valid.

3.3. Solving the Quintic Equation by a Numerical Approximation Method

To expand the Bring-Jerrard normal quintic equation (BJQ), $f(x) = x^5 + qx + r = 0$, into the two-factor form, Sage requires numeric values for *p and q* in order to solve it. We will follow the steps explained in section 2.3 for applying this method.

Next are details by using the facilities offered by the free open-source mathematics software system Sagemath (https://www.sagemath.org/). Let's start by expanding the equation BJQ into two factors, and solve the equations relating the coefficients.

```
var('a,b,c,d,e,f,g,h,i,j,k,l,x,y,p,q,r,s,t,u,v,A,B,C,D,E,F,G,H,T,K,L,M,N')

x^5 - 10*x + 2

expand((x^4 + A*x^3 + B*x^2 + C*x + D)*(x - E)).poly(x)

(A - E)*x^4 + x^5 - (A*E - B)*x^3 - (B*E - C)*x^2 - D*E - (C*E - D)*x

solve([A - E, A*E - B, B*E - C, C*E - D - 10, D*E + 2], A,B,C,D,E, solution_dict=true)

[{A: 1.724306472919419,

B: 2.973232669869595,

C: 5.126764621146644,

D: -1.159886628876292,

E: 1.724306472919419},

{A: -0.04961250951920438 - 1.781741795058672*I,

B: -3.172142423158106 + 0.1767933635362254*I,

C: 0.4723780710666982 + 5.643167572788117*I,
```

D: 0.03123165940893102 - 1.121627457212148*I,

E: -0.04961250951919667 - 1.781741795058679*I},

{A: -0.04961250951920438 + 1.781741795058672*I,

B: -3.172142423158106 - 0.1767933635362254*I,

C: 0.4723780710666982 - 5.643167572788117*I,

D: 0.03123165940893102 + 1.121627457212148*I,

E: -0.04961250951919667 + 1.781741795058679*I},

{A: -1.825113562621674,

B: 3.331039229181005,

C: -6.079524680073126,

D: 1.095822281167109,

E: -1.825113562621674},

{A: 0.2000320307495195,

B: 0.04001281142046625,

C: 0.008003843719970937,

D: -9.99839871897518,

E: 0.2000320307495195}]

As seen, this method resulted in a system of five equations with five unknowns. Using the Sage $solve()$ command to solve this system of equations, Sage requires, as mentioned in Equation (2.3), the numeric values of *q* and *r*. Fulfilling such a requirement, Sage gave us, in a very approximate way the values of the solutions. We have used for the values of *q* and *r* the Abel's concrete example, $x^5 - 10x + 2 = 0$, of an unsolvable quintic (irreducible and separable polynomial that its splitting field F and its Galois group G are not solvable, http://www.mathreference.com/fld-slv,xmp.html), whose expression is already in the Bring-Jerrard normal form (BJQ)

By observing only, the values of the variable E, we realize that they are, directly, the five roots of the quintic equation:

x1 = 1.724306472919419,

x2 = -0.04961250951919667 - 1.781741795058679*I

x3 = -0.04961250951919667 + 1.781741795058679*I

x4 = -1.825113562621674

x5 = 0.2000320307495195

Thus, we have solved by this simple approximate method a BJQ with numerical coefficients, not solvable. As it has been seen, with this approximate method we have obtained directly the roots of the quintic, and we have not needed to undo any process.

3.4. Solving the Sextic Equation Using the Numerical Approximation Method

For the sextic equation we expand the "Bring-Jerrard normal Sextic (BJSx)" equation, $e(x) = x^6 + px^2 + qx + r = 0$ (obtained by the original Bring's method) to the two-factor form, in order to solve it. We proceed following the same methodology.

var('a,b,c,d,e,f,g,h,i,j,k,l,x,y,p,q,r,s,t,u,v,A,B,C,D,E,F,G,H,T,K,L,M,N')

x^6 + x^2 + x + 4

expand((x^5 + A*x^4 + B*x^3 + C*x^2 + D*x + E)*(x - F)).poly(x)

(A - F)*x^5 + x^6 - (A*F - B)*x^4 - (B*F - C)*x^3 - (C*F - D)*x^2 - E*F - (D*F - E)*x

solve([(A - F), -(A*F - B), -(B*F - C), -(C*F - D) - 1, -E*F - 4, -(D*F - E) - 1], A,B,C,D,E,F, solution_dict=true)

[{A: 1.130052386159644 + 0.7571415287366167*I,

B: 0.7037551009274823 + 1.711219182418748*I,

C: -0.5003549767048322 + 2.466609533306887*I,

D: -1.432999848196437 + 2.408558456864301*I,

E: -2.442984529902534 + 1.636813535641703*I,

F: 1.130052386159644 + 0.7571415287366167*I},

{A: 1.130052386159644 - 0.7571415287366167*I,

B: 0.7037551009274823 - 1.711219182418748*I,

C: -0.5003549767048322 - 2.466609533306887*I,

D: -1.432999848196437 - 2.408558456864301*I,

E: -2.442984529902534 - 1.636813535641703*I,

F: 1.130052386159644 - 0.7571415287366167*I},

{A: -1.055849669379472 + 0.6571572389843788*I,

B: 0.682962887579167 - 1.387718507023962*I,

C: 0.1908431235144778 + 1.914036132363349*I,

D: -0.4593243489263448 - 1.895520477408277*I,

E: 2.730632465323797 + 1.699536348434814*I,

F: -1.055849669379472 + 0.6571572389843788*I},

{A: -1.055849669379472 - 0.6571572389843788*I,

B: 0.682962887579167 + 1.387718507023962*I,

C: 0.1908431235144778 - 1.914036132363349*I,

D: -0.4593243489263448 + 1.895520477408277*I,

E: 2.730632465323797 - 1.699536348434814*I,

F: -1.055849669379472 - 0.6571572389843788*I},

{A: -0.07420271678017132 + 1.179925434798408*I,

B: -1.386717988506648 - 0.1751073457201374*I,

C: 0.3095118531903559 - 1.623230384750881*I,

D: 2.89232419712278 + 0.4856490124595297*I,

E: 0.2123520645787341 + 3.376690409741947*I,

F: -0.07420271678017135 + 1.179925434798407*I},

{A: -0.07420271678017132 - 1.179925434798408*I,

B: -1.386717988506648 + 0.1751073457201374*I,

C: 0.3095118531903559 + 1.623230384750881*I,

D: 2.89232419712278 - 0.4856490124595297*I,

E: 0.2123520645787341 - 3.376690409741947*I,

F: -0.07420271678017135 - 1.179925434798407*I}]

The roots of the sextic, corresponding to the values of F, are then:

x1 = 1.130052386159644 + 0.7571415287366167*I,

x2 = 1.130052386159644 - 0.7571415287366167*I,

x3 = -1.055849669379472 + 0.6571572389843788*I,

x4 = -1.055849669379472 - 0.6571572389843788*I,

x5 = -0.07420271678017135 + 1.179925434798407*I},

x6 = -0.07420271678017135 - 1.179925434798407*I

3.5. Solving the Septic

Starting from the normal septic equation (BJSp), $d(z) = z^7 + Z_3 z^3 + Z_2 z^2 + Z_1 z + Z_0 = 0$, without the 2nd, 3rd, and 4th term, and applying the same strategy of using numerical coefficients and expanding the equation into the two-factor form, we obtain:

var('a,b,c,d,e,f,g,h,i,j,k,l,x,y,p,q,r,s,t,u,v,A,B,C,D,E,F,G,H,T,K,L,M,N')

x^7 + x^3 + x^2 + x + 2

Expand: ((x^6 + A*x^5 + B*x^4 + C*x^3 + D*x^2 + E*x + F)*(x - G)).poly(x)

(A - G)*x^6 + x^7 - (A*G - B)*x^5 - (B*G - C)*x^4 - (C*G - D)*x^3 - (D*G - E)*x^2 - F*G - (E*G - F)*x

Solve: ([(A - G), - (A*G - B), - (B*G - C), - (C*G - D) - 1, - (D*G - E) - 1, - F*G - 1, - (E*G - F) - 1], A,B,C,D,E,F,G, solution_dict=true)

Whose roots are those of the values of G in the seven solution sets, respectively:

x1 = 1.021266493741164 - 0.7042491644258739*I,

x2 = 1.021266493741164 + 0.7042491644258739*I,

x3 = -0.8342155919153031,

x4 = -0.6649220840565686 - 0.7676211763057554*I,

x5 = -0.6649220840565686 + 0.7676211763057554*I,

x6 = 0.06076337020943194 - 0.8669183701205007*I,

x7 = 0.06076337020943194 + 0.8669183701205007*I

Thus, using this simple method we can solve equations of 8th, 9th, …, 30th degree and obtain their roots. For example, making the equation of 30th degree:

var('a,b,c,d,e,f,g,h,i,j,k,l,x,y,p,q,r,s,t,u,v,A,B,C,D,E,F,G,H,J,K,L,M,N,P,Q,R,S,T,V,W')

Expand: ((x^29 + a*x^28 + b*x^27 + c*x^26 + d*x^25 + e*x^24 + f*x^23 + g*x^22 + h*x^21 + j*x^20 + k*x^19 + A*x^18 + B*x^17 + C*x^16 + D*x^15 + E*x^14 + F*x^13 + G*x^12 + H*x^11 + J*x^10 + K*x^9 + L*x^8 + M*x^7 + N*x^6 + P*x^5 + Q*x^4 + R*x^3 + S*x^2 + T*x + V)*(x - W)).poly(x)

-(W - a)*x^29 + x^30 - (W*a - b)*x^28 - (W*b - c)*x^27 - (W*c - d)*x^26 - (W*d - e)*x^25 - (W*e - f)*x^24 - (W*f - g)*x^23 - (W*g - h)*x^22 - (W*h - j)*x^21 - (W*j - k)*x^20 - (W*k - A)*x^19 - (A*W - B)*x^18 - (B*W - C)*x^17 - (C*W - D)*x^16 - (D*W - E)*x^15 - (E*W - F)*x^14 - (F*W - G)*x^13 - (G*W - H)*x^12 - (H*W - J)*x^11 - (J*W - K)*x^10 - (K*W - L)*x^9 - (L*W - M)*x^8 - (M*W - N)*x^7 - (N*W - P)*x^6 - (P*W - Q)*x^5 - (Q*W - R)*x^4 - (R*W - S)*x^3 - (S*W - T)*x^2 - V*W - (T*W - V)*x

Solve: ([-(W - a), - (W*a - b), - (W*b - c), - (W*c - d), - (W*d - e) - 1, - (W*e - f) - 1, - (W*f - g) - 1, - (W*g - h) - 1, - (W*h - j) - 1, - (W*j - k) - 1, - (W*k - A) - 1, - (A*W - B) - 1, - (B*W - C) - 1, - (C*W - D) - 1, - (D*W - E) - 1, - (E*W - F) - 1, - (F*W - G) - 1, - (G*W - H) - 1, - (H*W - J) - 1, - (J*W - K) - 1, - (K*W - L) - 1, - (L*W - M) - 1, - (M*W - N) - 1, - (N*W - P) - 1, - (P*W - Q) - 1, - (Q*W - R) - 1, - (R*W - S) - 1, - (S*W - T) - 1, - V*W - 1, - (T*W - V) - 1], a,b,c,d,e,f,g,h,j,k,A,B,C,D,E,F,G,H,J,K,L,M, N,P,Q,R,S,T,V,W, solution_dict=true)

Whose roots, corresponding to the values of W, respectively, are:

x1 = 0.5506606400848768 + 0.8073153228833507*I,

x2 = 0.5506606400848768 - 0.8073153228833507*I,

x3 = 0.3580285801177567 + 0.9077217657205909*I,

x4 = 0.3580285801177567 - 0.9077217657205909*I,

x5 = 0.1536948728161841 + 0.9724452224800839*I,

x6 = 0.1536948728161841 - 0.9724452224800839*I,

x7 = 0.7266903938964572 + 0.6706434890048041*I,

x8 = 0.7266903938964572 - 0.6706434890048041*I,

x9 = -0.04888315358340183 + 1.005348461862035*I,

x10 = -0.04888315358340183 - 1.005348461862035*I,

x11 = -0.7232207831943749 + 0.6375866171844605*I,

x12 = -0.7232207831943749 - 0.6375866171844605*I,

x13 = -0.3982750849513918 + 0.9025610347614539*I,

x14 = -0.3982750849513918 - 0.9025610347614539*I,

x15 = 0.8898549009751727 + 0.4898388398467622*I,

x16 = 0.8898549009751727 - 0.4898388398467622*I,

x17 = -0.2232933265289773 + 0.9883252076235574*I,

x18 = -0.2232933265289773 - 0.9883252076235574*I,

x19 = -0.8460182376820754 + 0.467254678185422*I,

x20 = -0.8460182376820754 - 0.467254678185422*I,

x21 = -0.5714801906395764 + 0.783792279906856*I,

x22 = -0.5714801906395764 - 0.783792279906856*I,

x23 = 0.9807028294251598 + 0.2353613383980424*I,

x24 = 0.9807028294251598 - 0.2353613383980424*I,

x25 = -0.9916497607907169 + 0.09053664110983789*I,

x26 = -0.9916497607907169 - 0.09053664110983789*I,

x27 = -0.9365714807184794 + 0.2800602879204082*I,

x28 = -0.9365714807184794 - 0.2800602879204082*I,

x29 = 1.079759800773415 + 0.4949815792672019*I,

x30 = 1.079759800773415 - 0.4949815792672019*I

4. Final Comments and Conclusion

The approximate method of calculating the coefficients of any equation by its reduction to the product of another equation of degree *n*-1, by another of degree 1, for *n* equal to 5 or greater, as it was demonstrated even for *n* equal to 30, applying Sage, can be used to obtain directly the roots of such equation.

Given the invariant form of the approximate method used to solve a wide range of equations of degree 5 or higher, with a high degree of approximation, it can be easily automated in Sage.

We have observed that Sage's ability to approximate solution sets for equations of order greater than 30 is limited, although it is not that limited! since, in general, equations of such high degrees are rarely presented. The referred limitation can be easily overcome by working in particular on the approximation algorithms used in Sage, and of course on Sage's own capability. Taking into account that multiple roots are not calculated from each other by the used command: solve([EqsOfCoeff], variables, solution_dict=true), as for example, see that for solving the equation x^20 + x^16 + x^15 + x^14 + x^13 + x^12 + x^11 + x^10 + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0, by using our method (section 2.3), we have only 16 solution sets.

Expand: ((x^19 + a*x^18 + b*x^17 + c*x^16 + d*x^15 + e*x^14 + f*x^13 + g*x^12 + h*x^11 + j*x^10 + A*x^9 + B*x^8 + C*x^7 + D*x^6 + E*x^5 + F*x^4 + G*x^3 + H*x^2 + J*x + K)*(x - W)).poly(x)

Solve: ([-(W - a), - (W*a - b), - (W*b - c), - (W*c - d) -1, - (W*d - e) -1, - (W*e - f) -1, - (W*f - g) -1, - (W*g - h) -1, - (W*h - j) -1, - (W*j - A) -1, - (A*W - B) -1, - (B*W - C) -1, - (C*W - D) -1, - (D*W - E) -1, - (E*W - F) -1, - (F*W - G) -1, - (G*W - H) -1, - (H*W - J) -1, - K*W -1, - (J*W - K) -1], a,b,c,d,e,f,g,h,j,A,B,C,D,E,F,G,H,J,K,W, solution_dict=true)

To have all twenty sets and in general, to have the complete roots of any original equation directly, you must use instead of the solve command, commands similar to:

R.<a,b,c,d,e,f,g,h,j,A,B,C,D,E,F,G,H,J,K,W,x> = QQ[]

I = R.ideal([-(W - a), - (W*a - b), - (W*b - c), - (W*c - d) -1, - (W*d - e) -1, - (W*e - f) -1, - (W*f - g) -1, - (W*g - h) -1, - (W*h - j) -1, - (W*j - A) -1, - (A*W - B) -1, - (B*W - C) -1, - (C*W - D) -1, - (D*W - E) -1, - (E*W - F) -1, - (F*W - G) -1, - (G*W - H) -1, - (H*W - J) -1, - K*W -1, - (J*W - K) -1])

F_W = I.elimination_ideal([a,b,c,d,e,f,g,h,j,A,B,C,D,E,F,G,H,J,K,x]).gen(0).polynomial(W); F_W

F_W.roots(QQbar).

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