https://doi.org/10.51483/IJPAMR.4.1.2024.28-33 ISSN: 2789-9160

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Publisher's Home Page: https://www.svedbergopen.com/

Research Paper Open Access

Gocgen Approach for Zeta Function in Twin Primes

Ahmet F. Gocgen[1*]

[1]Independent Researcher, Turkey. E-mail: ahmetfgocgen@gmail.com

Article Info

Volume 4, Issue 1, April 2024
Received : 27 January 2024
Accepted : 21 March 2024
Published : 05 April 2024
doi: 10.51483/IJPAMR.4.1.2024.28-33

Abstract

I had previously developed an approach called Gocgen approach, which claimed to prove twin prime conjecture. In this paper, I processed the previously developed Gocgen approach with the zeta function and explained the relationship between the zeta function with some formulas to offer another perspective on the zeta function, and also created a new function that explains the Gocgen approach using the zeta function. Understanding the new function created will provide insight into the frequency of prime numbers.

Keywords: *Number theory, Twin prime conjecture, Gocgen approach, Prime numbers, Zeta function*

1. Introduction

Let p_n denote the n^{th} prime. Twin prime conjecture is conjectured that

$$\liminf_{n\to\infty}\left(p_{n+1}-p_n\right)=2$$

Lemma 1

According to Aysun and Gocgen (Aysun and Gocgen, 2023):

$n_p + p$ gives all composite numbers where n is a positive natural numbers and p is a prime number.

Proof: $n_p + p = p(n+1)$. Then, according to fundamental theorem of arithmetic:

$(n+1 \in \mathbb{C}) \oplus (n+1 \in \mathbb{P})$

Let $n+1 \in \mathbb{C}$:

$n+1 = p_m \times \ldots \times p_{m+k}$

Then,

$p(n+1) = \mathrm{p} \times (p_m \times \ldots \times p_{m+k})$

* Corresponding author: Ahmet F. Gocgen, Independent Researcher, Turkey. E-mail: ahmetfgocgen@gmail.com

Let $n + 1 \in \mathbb{P}$:

$n + 1 = p_m$

Then,

$p(n + 1) = p \times p_m$

Lemma 2

According to Aysun and Gocgen (Aysun and Gocgen, 2023).

$2np + p$ gives all odd composite numbers where n is a positive natural numbers and p is an odd prime numbers.

Proof: $np + p$ gives odd composite numbers where p is a odd number and n is a even number. Then as already proved $np + p$ gives all composite numbers where n is a positive natural number and p is an prime number. Only possibility for odd composite just specified. Therefore, $np + p$ gives all odd composite numbers where p is a odd number and n is a even number. This equal to: $2np + p$ gives all odd composite numbers where n is a positive natural numbers and p is an odd prime numbers.

Lemma 3

According to Rhafli (Rhafli, 2019):

$2np + p^2$ gives all odd composite numbers where n is a natural numbers and p is a odd prime numbers.

Proof: with $n \in \mathbb{N}$ and p are all the primes except 2 which satisfy $p \le \sqrt{N}$, the equation $2np + p^2 =$ all odd composite is true since if we divide it by p we get the trivial equation for odd numbers. For a given interval $I = [a, b]$ one calculates the constant n and iterates to generates the odd composites included in the interval I.

Since the proofs of the following statements are dense and long, only the statements accepted as Lemma are given without citing any evidence, by citing articles directly related to the proof.

Lemma 4

According to Zhang, Maynard and Polymath project (Zhang, 2014; Polymath, 2014; Hasanalizade, 2012):

$$\liminf_{n\to\infty}\left(p_{n+1} - p_n\right) \le 7\times 10^7,$$

$$\liminf_{n\to\infty}\left(p_{n+1} - p_n\right) \le 4680,$$

$$\liminf_{n\to\infty}\left(p_{n+1} - p_n\right) \le 246.$$

Lemma 5

According to Gocgen (Gocgen, 2024):

The expressions $6n + p_1 \times ... \times p_k - 4$ and $6n + p_1 \times ... \times p_k - 2$ produce composite numbers

that cannot be divided by $p_1 ... p_k$ primes, that is, cannot be expressed with $6n + p_1 \times ... \times p_k - 6$.

$6n + p_1 \times ... \times p_k - 4$, $6n + p_1 \times ... \times p_k - 2$ (s. group)

$6n + p_1 \times ... \times p_k + 2$, $6n + p_1 \times ... \times p_k + 4$ (s + 1. group)

Accordingly, let's examine the possibilities in which at least one value in both groups is a composite, and let's look at the gap that must remain between the composites forever after a certain number so that the twin primes are not infinite:

1) $6n + p_1 \times ... \times p_k - 4$ and $6n + p_1 \times ... \times p_k + 2$ can be composite. Bounded of gaps: 6.
2) $6n + p_1 \times ... \times p_k - 4$ and $6n + p_1 \times ... \times p_k + 4$ can be composite. Bounded of gaps: 8.
3) $6n + p_1 \times ... \times p_k - 2$ and $6n + p_1 \times ... \times p_k + 2$ can be composite. Bounded of gaps: 4.

4) $6n + p_1 \times ... \times p_k - 2$ and $6n + p_1 \times ... \times p_k + 4$ can be composite. Bounded of gaps: 6.
5) $6n + p_1 \times ... \times p_k - 4$, $6n + p_1 \times ... \times p_k - 2$ and $6_n + p_1 \times ... \times p_k + 2$, $6n + p_1 \times ... \times p_k + 4$ can be composite. Bounded of gaps: 2.

Then we can pose a new question as follows: Can the gaps between odd composite numbers that are not divisible by $p_1, ..., p_k$ be 6 and/or 8 and/or 4 and/or 2 forever after a certain number?

Therefore, the following question arises:

$p \neq 2c$: composite numbers that cannot be divided by $p_1, ..., p_k$:

$$\liminf_{n\to\infty} \left(c_{n+1} - c_n\right) \geq 10?$$

For the difference between composite numbers that cannot be divided by $p_1, ..., p_k$:

$$\lim_{n\to 1} \left(c_{n+1} - c_n\right) = \left(\left(p_{k+2} \cdot p_{k+1}\right) - p_{k+1}^2\right)$$

Since p is a prime number, and

$$\liminf_{n\to\infty} \left(p_{n+1} - p_n\right) \leq 246$$

has been proven, the number of cases where p_{k+1} and p_{k+2} differ by 246 is infinite. Therefore ($p_{k+1} = x, p_{k+2} = x + 246$),

$$\liminf_{n\to\infty} \left(c_{n+1} - c_n\right) = \left(\left(x + 246 \cdot x\right) - x^2\right)$$

Let's edit this expression:

$$\liminf_{n\to\infty} \left(c_{n+1} - c_n\right) = \left(x^2 + 246x - x^2\right)$$

$$\liminf_{n\to\infty} \left(c_{n+1} - c_n\right) = \left(246x\right)$$

When it is not forgotten that x is prime:

$$\liminf_{n\to\infty} \left(c_{n+1} - c_n\right) \geq 10.$$

Accordingly, the gap between odd composite numbers that cannot be divided by $p_1, ..., p_k$ cannot be 6 and/or 8 and/or 4 and/or 2 forever after a certain number.

Therefore twin primes are infinite.

2. Theorems and Proofs

Let's apply the same method to the zeta function as Marouane:

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \dots$$

$$\frac{1}{2^s}\zeta(s) = \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \frac{1}{10^s} + \dots$$

$$\left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} + \frac{1}{11^s} + \dots$$

$$\left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 + \sum_{n=1}^{\infty} \frac{1}{\left(p_n\right)^s} + \sum_{n=1}^{\infty} \frac{1}{\left(p_n + 2p_n c\right)^s}$$

$$\zeta(s)=\left(1-\frac{1}{2^s}\right)^{-1}\left(1+\sum_{n=1}^{\infty}\frac{1}{(p_n)^s}+\sum_{n=1}^{\infty}\frac{1}{(p_n+2p_nc)^s}\right)$$

To set the zeta function as divisible by a specific prime:

$$\left[p_a\cdot\left(1-\frac{1}{2^s}\right)\right]\zeta(s)=p_a+\frac{1}{(p_a\cdot 3)^s}+\frac{1}{(p_a\cdot 5)^s}+\frac{1}{(p_a\cdot 7)^s}+\ldots$$

$$\left[p_a\cdot\left(1-\frac{1}{2^s}\right)\right]\zeta(s)=p_a+\sum_{n=1}^{\infty}\frac{1}{(p_a\cdot p_n)^s}+\sum_{n=1}^{\infty}\frac{1}{(p_a\cdot(p_n+2p_nc))^s}$$

$$\zeta(s)=\left[p_a\cdot\left(1-\frac{1}{2^s}\right)\right]^{-1}\left(p_a+\sum_{n=1}^{\infty}\frac{1}{(p_a\cdot p_n)^s}+\sum_{n=1}^{\infty}\frac{1}{(p_a\cdot(p_n+2p_nc))^s}\right)$$

To set the same operation to the zeta function as divisible by more than one prime:

$$\left[p_a\cdot p_{a+k}\cdot\left(1-\frac{1}{2^s}\right)\right]\zeta(s)=p_a\cdot p_{a+k}+\frac{1}{(p_a\cdot p_{a+k}\cdot 3)^s}+\frac{1}{(p_a\cdot p_{a+k}\cdot 5)^s}+\frac{1}{(p_a\cdot p_{a+k}\cdot 7)^s}+\ldots$$

$$\left[p_a\cdot p_{a+k}\cdot\left(1-\frac{1}{2^s}\right)\right]\zeta(s)=p_a\cdot p_{a+k}+\sum_{n=1}^{\infty}\frac{1}{(p_a\cdot p_{a+k}\cdot p_n)^s}+\sum_{n=1}^{\infty}\frac{1}{(p_a\cdot p_{a+k}\cdot(p_n+2p_nc))^s}$$

$$\zeta(s)=\left[p_a\cdot p_{a+k}\cdot\left(1-\frac{1}{2^s}\right)\right]^{-1}\left(p_a\cdot p_{a+k}+\sum_{n=1}^{\infty}\frac{1}{(p_a\cdot p_{a+k}\cdot p_n)^s}+\sum_{n=1}^{\infty}\frac{1}{(p_a\cdot p_{a+k}\cdot(p_n+2p_nc))^s}\right)$$

For those divisible by both primes:

$$\zeta(s)=\left[p_a\cdot p_{a+k}\cdot\left(1-\frac{1}{2^s}\right)\right]^{-1}\left(p_a\cdot p_{a+k}+\sum_{n=1}^{\infty}\frac{1}{(p_a\cdot p_{a+k}\cdot p_n)^s}+\sum_{n=1}^{\infty}\frac{1}{(p_a\cdot p_{a+k}\cdot(p_n+2p_nc))^s}\right)$$

For those divisible only by the first prime:

$$\left[p_a\cdot\left(1-\frac{1}{2^s}\right)\right]^{-1}\left(p_a+\sum_{n=1}^{\infty}\frac{1}{(p_a\cdot p_n)^s}+\sum_{n=1}^{\infty}\frac{1}{(p_a\cdot(p_n+2p_nc))^s}\right)$$

For those divisible only by the second prime:

$$\left[p_{a+k}\cdot\left(1-\frac{1}{2^s}\right)\right]^{-1}\left(p_{a+k}+\sum_{n=1}^{\infty}\frac{1}{(p_{a+k}\cdot p_n)^s}+\sum_{n=1}^{\infty}\frac{1}{(p_{a+k}\cdot(p_n+2p_nc))^s}\right)$$

If we write the expression steady:

$$\zeta(s)=\left[p_a\cdot p_{a+k}\cdot\left(1-\frac{1}{2^s}\right)\right]^{-1}\left(p_a\cdot p_{a+k}+\sum_{n=1}^{\infty}\frac{1}{(p_a\cdot p_{a+k}\cdot p_n)^s}+\sum_{n=1}^{\infty}\frac{1}{(p_a\cdot p_{a+k}\cdot(p_n+2p_nc))^s}\right)$$

$$\left[p_a \cdot \left(1-\frac{1}{2^s}\right)\right]^{-1}\left(p_a + \sum_{n=1}^{\infty}\frac{1}{\left(p_a \cdot p_n\right)^s} + \sum_{n=1}^{\infty}\frac{1}{\left(p_a \cdot \left(p_n + 2p_n c\right)\right)^s}\right)$$

$$\left[p_{a+k} \cdot \left(1-\frac{1}{2^s}\right)\right]^{-1}\left(p_{a+k} + \sum_{n=1}^{\infty}\frac{1}{\left(p_{a+k} \cdot p_n\right)^s} + \sum_{n=1}^{\infty}\frac{1}{\left(p_{a+k} \cdot \left(p_n + 2p_n c\right)\right)^s}\right)$$

For odd numbers that cannot be divided by the first and second prime (p_n must be greater then p_{a+k+1}):

$$\zeta(s) = \left[p_{a+k+1} \cdot \left(1-\frac{1}{2^s}\right)\right]^{-1}\left(p_{a+k+1}^2 + \frac{1}{\left(p_{a+k+1} \cdot p_{a+k+2}\right)^s} + \sum_{n=1}^{\infty}\frac{1}{\left(p_{a+k+1} \cdot p_n\right)^s} + \sum_{n=1}^{\infty}\frac{1}{\left(p_{a+k+1} \cdot \left(p_n + 2p_n c\right)\right)^s}\right)$$

Now let's rearrange this expression according to Lemma 4:

$$\zeta(s) = \left[p_{a+k+1} \cdot \left(1-\frac{1}{2^s}\right)\right]^{-1}\left(x^2 + \frac{1}{\left(x \cdot 246x\right)^s} + \sum_{n=1}^{\infty}\frac{1}{\left(x \cdot p_n\right)^s} + \sum_{n=1}^{\infty}\frac{1}{\left(x \cdot \left(p_n + 2p_n c\right)\right)^s}\right)$$

In this regard, let's create another function based on the zeta function to simplify our operations:

$$\varepsilon = x \times x + 246x \times x + \sum_{n=1}^{\infty}\frac{1}{\left(p_{a+k+1} \cdot p_n\right)^s} + \sum_{n=1}^{\infty}\frac{1}{\left(p_{a+k+1} \cdot \left(p_n + 2p_n c\right)\right)^s}$$

$$\varepsilon = x^2 + 246x^2 + \sum_{n=1}^{\infty}\frac{1}{\left(p_{a+k+1} \cdot p_n\right)^s} + \sum_{n=1}^{\infty}\frac{1}{\left(p_{a+k+1} \cdot \left(p_n + 2p_n c\right)\right)^s}$$

For the difference between odd numbers that are not divisible by the specified primes:

$$\varepsilon = \left|x^2 - 246x^2\right| + \sum_{m=l}^{\infty}\left|\sum_{n=m}^{\infty}\frac{1}{\left(p_{a+k+1} \cdot p_n\right)^s} \oplus \sum_{n=m}^{m+1}\frac{1}{\left(p_{a+k+1} \cdot \left(p_n + 2p_n c\right)\right)^s}\right.$$

$$\left. - \sum_{n=m+1}^{\infty}\frac{1}{\left(p_{a+k+1} \cdot p_n\right)^s} \oplus \sum_{n=m+1}^{m+2}\frac{1}{\left(p_{a+k+1} \cdot \left(p_n + 2p_n c\right)\right)^s}\right|$$

The first part to the right of the equation is the part where there is a 246x difference between the numbers that are not divisible by the specified primes. Considering that there are an infinite number of situations with 216 differences between them (Lemma 4), there will be a situation with a 216x difference between the only primes that cannot be divided into an infinite number of specified primes with different x values. This shows the infinity of twin primes, as in Lemma 5, based on the fact that the difference between odd numbers that cannot be divided by certain primes is greater than 10. In addition, by understanding the part on the second side of the equation, it is possible to obtain new information about the frequencies between primes.

Funding

This research received no external funding.

Acknowledgments

I would like to express my endless gratitude to Marouane Rhafli for inspiring this work.

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Cite this article as: Ahmet F. Gocgen (2024). Gocgen Approach for Zeta Function in Twin Primes. *International Journal of Pure and Applied Mathematics Research*, 4(1), 28-33. doi: 10.51483/IJPAMR.4.1.2024.28-33.