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# The Collatz Conjecture: A New Proof Using Algebraic Inverse Tree

Eduardo Diedrich[1*]

[1]Independent Researcher, Graduated from Universidad Nacional de Salta, Argentina. E-mail: eduardo.diedrich@outlook.com.ar

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## Abstract

The Collatz Conjecture has intrigued mathematicians for decades. It proposes that iterating the function $C(n) = n/2$ for even $n$ or $C(n) = 3n + 1$ for odd n on any natural number ultimately leads to the cycle 1, 2, 4. Despite its simplicity, the conjecture's unpredictable nature complicates proofs. Algebraic Inverse Trees (AITs) offer a novel approach by modeling Collatz sequences in reverse. AITs recursively track numeric pathways using $C^{-1}$, enabling global analysis, anomaly detection, and convergence estimation. They exhibit topological equivalence with Collatz sequences, allowing for the transfer of key properties. By establishing mappings between AITs and Collatz sequences, discrete systems can exchange cardinal attributes, facilitating convergence proofs. This multidimensional strategy combines inverse algebraic models and topological equivalences for an innovative approach to this historic puzzle.

***Keywords:*** *AIT, Topology, Transport, Homeomorphism, Reverse collatz function*



## 1. Introduction

The Collatz Conjecture is a famous unsolved problem in mathematics stating that starting from any positive integer $n$, iterating the function.

$$C(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ 3n+1 & \text{if } n \text{ is odd} \end{cases} \quad ...(1)$$

will eventually reach the number 1, at which point the sequence enters the trivial cycle $1 \to 4 \to 2 \to 1$. Despite the simple formulation, the trajectory of the iteration appears erratic, and no proof exists for all starting values. The Collatz conjecture asserts that, regardless of the starting number, the Collatz sequence will always converge to 1. This conjecture has been verified for all initial numbers up to $2^{68}$, but it has not yet been proven.

In this paper, we introduce AIT (AITs) as a novel representation for inversely modeling the relationships inherent in Collatz sequences. By recursively building inverted tree rooted at 1 using the inverse Collatz function $C^{-1}$, the structure of AITs formally captures all possible convergence pathways to 1 from any starting natural number.

* *Corresponding author: Eduardo Diedrich, Independent Researcher, Graduated from Universidad Nacional de Salta, Argentina. E-mail: eduardo.diedrich@outlook.com.ar*

We establish key properties of AITs, including guaranteed path convergence and absence of non-trivial cycles. Furthermore, we prove a topological equivalence between the space of AITs and the space of Collatz sequences. Leveraging this equivalence, convergence transfers from AIT paths to Collatz sequences, indicating that trajectories from all natural numbers provably approach 1.

The AIT perspective thus provides new structural insights and a platform for formally reasoning about convergence in the context of the infamous yet evasive Collatz Conjecture.

## 2. Foundational Framework

**Definition 2.1 (Topology):** Let $X$ be a set. A topology $\tau$ on $X$ consists of a family of subsets of $X$, called open sets, that satisfies:

1. $\phi \in \tau$ and $X \in \tau$
2. Any union of open sets is open.
3. Any finite intersection of open sets is open.

Then the ordered pair $(X, \tau)$ constitutes a topological space. Additionally, the following concepts are defined:

- Closed sets as the complements of the open sets.
- A basis for the topology $\tau$ if the family of open sets generated by the basis equals $\tau$.
- A subbasis for $\tau$ if the family of open sets generated by the finite intersections of the subbasis equals $\tau$.

**Definition 2.2:** Let $(X, \tau)$ be a topological space. We define the following concepts precisely:

- Compatibility: For any $B, C \in \tau, B \cap C \in \tau$.

  Formally: $\forall B \forall C[(B \in \tau \wedge C \in \tau) \rightarrow (B \cap C) \in \tau]$
- Completeness: For every convergent sequence $(x_n)$ in $X$, there exists $x \in X$ such that $\lim(x_n) = x$.

  Formally: $\forall (x_n) \subseteq X \forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n (n > N \rightarrow d(x_n, x) < \varepsilon) \rightarrow \exists x \in X \forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n (n > N \rightarrow d(x_n, x) < \varepsilon)$
- Continuity: For every function $f{:}(X, \tau) \rightarrow (Y, \theta)$ and every open subset $V$ of $Y, f^{-1}(V)$ is an open subset of $X$. Formally: $\forall V((V \in \theta) \rightarrow (f^{-1}(V) \in \tau))$

**Definition 2.3 (Topological Transport):** Let $f: X \rightarrow Y$ be a homeomorphism between topological spaces. The topological transport through $f$ is defined as the mechanism by which any topological property $P$ invariant under homeomorphisms and demonstrated in $X$ is preserved and transferred to $Y$ by the homeomorphic action.

Formally, for all $P$ such that:

1. $P$ is a topological property,
2. $P$ holds in $X$,
3. $P$ is invariant under homeomorphisms,

it follows that $f$ transfers $P$ from $X$ to $Y$, that is, $P$ holds in $Y$ by topological transport through $f$.

**Definition 2.4 (Collatz Function):** Let $\mathbb{N}$ be the set of natural numbers. The Collatz function $C$: $\mathbb{N} \rightarrow \mathbb{N}$ is defined for all $n \in \mathbb{N}$ as follows:

$$C(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ 3n + 1 & \text{if } n \text{ is odd.} \end{cases}$$

This function $C$ is the well-known Collatz function, which, according to the conjecture bearing its name, when iterated from any natural number, will eventually reach the trivial cycle $\{1, 2, 4\}$.

**Definition 2.5 (Inverse Collatz Function):** Let $N$ be the set of natural numbers. The multi-valued inverse Collatz function $C^{-1} : N \rightarrow \mathcal{P}(N)$ is defined for all $n \in N$ as:

$$C^{-1}(n) = \begin{cases} \{2n\} & \text{if } n \not\equiv 4 \pmod{6} \\ \left\{2n, \dfrac{n-1}{3}\right\} & \text{if } n \equiv 4 \pmod{6} \end{cases}$$

where $\mathcal{P}(N)$ denotes the power set of $N$.

*Properties 1:* The function $C^{-1}(n)$ satisfies the following properties:

- Non-emptiness: $C^{-1}(n) \neq \phi$ for all $n \in N$
- Injectivity: For all $x, y \in N$, if $C(x) = C(y) = n$, then $x, y \in C^{-1}(n)$
- Surjectivity: For all $n \in N$, there exists $x \in N$ such that $C^{-1}(x) = n$

**Definition 2.6:** Given integers $a$ and $n$, we say that $a$ is congruent with $b$ modulo $n$, denoted as $a \equiv b \pmod{n}$, if $n$ divides the difference $a - b$.

*Properties 2:* Modular congruences modulo $n$ satisfy the following properties:

1. Reflexivity: $a \equiv a \pmod{n}$
2. Symmetry: If $a \equiv b \pmod{n}$, then $b \equiv a \pmod{n}$
3. Transitivity: If $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$, then $a \equiv c \pmod{n}$

Since the inverse function $C^{-1}$ is defined by cases according to the congruence modulo 6, these properties guarantee a unique and consistent partition of the domain.

Furthermore, it is demonstrated that the relation of congruence modulo $n$ is an equivalence relation and generates equivalence classes of numbers that behave the same with respect to the modulus. This allows for a combined treatment of the classes.

Therefore, the modularity of the definition of $C^{-1}$ is perfectly valid, as it is based on the solid properties of congruences that ensure a proper partition and grouping of natural numbers into equivalence classes modulo 6.

### *2.1. Properties of the Collatz Function*

**Theorem 2.1 (Determinism of *C*):** Let $C: \mathbb{N} \to \mathbb{N}$ be the Collatz function defined as:

$$C(n) = \begin{cases} \dfrac{n}{2} & \text{if } n \text{ is even} \\ 3n+1 & \text{if } n \text{ is odd} \end{cases}$$

Then, given an input $n \in \mathbb{N}$ and a sequence $C^k(n)$ obtained by iterating $C: \forall n \in \mathbb{N}, \forall k \in \mathbb{N}, \exists! C^k(n)$

In other words, $C$ always generates a unique sequence of values for any initial input $n$.

**Proof:** We prove this by mathematical induction on $\mathbb{N}$.

Base case: Let $n = 1$. Then: $C(1) = 4C^2(1) = 2C^3(1) = 1$

The sequence is unique and deterministic.

Inductive hypothesis: Suppose that $\forall n < k, \exists! \langle C(n), C^2(n), \ldots \rangle$.

Inductive step: Let $n = k + 1$. Then:

- If $k + 1$ is even, $C(k + 1)$ is unique by definition.
- If $k + 1$ is odd, $C(k + 1)$ is also unique.

In both cases, by the IH and the definition of $C$, $\exists! C^{k+1}(n)$.

By PMI, $\forall n \in \mathbb{N}, \exists! \langle C(n), C^2(n), \ldots \rangle$.

**Theorem 2.2:** Let $C: \mathbb{N} \to \mathbb{N}$ be the Collatz function. Let $C^{-1}: \mathbb{N} \to \mathcal{P}(\mathbb{N})$ be its multivalued inverse.

Then, there exists an injective correspondence between any direct sequence $S_d$ generated by iterations of $C$ and any inverse sequence $S_i$ generated by iterations of $C^{-1}$.

Formally: $\forall S_d = \langle s_1, s_2, \ldots, s_n \rangle, \forall S_i = \langle s_1', s_2', \ldots, s_m' \rangle, \exists!\, g: S_d \to S_i$

Where $g$ is an injective function that correlates each term of the direct sequence $S_d$ with one and only one term of the inverse $S_i$.

**Proof:** Define $g: S_d \to S_i$ as:

$g(s_k) = s_k'$ if and only if $s_k'$ is a pre-image of $s_k$ under $C$.

Since $C$ is deterministic (injective), each $s_k \in S_d$ is associated with a unique image in $S_i$. Thus, $g$ is well-defined.

Moreover, since $|S_d| = |S_i|$, $g$ establishes a one-to-one injective correspondence between both sequences.

Therefore, the existence of a bijection between any direct and inverse sequence is proven.

**Theorem 2.3:** Let $C: \mathbb{N} \to \mathbb{N}$ be the Collatz function. Then:

$\forall n \in \{1, 2, 4\}, C^3(n) = n$

Moreover, this is the only cycle of length 3. In other words:

$\nexists m \notin \{1,2,4\}: C^3(m) = m$

**Proof:** By direct evaluation of $C$:

$C^3(1) = C(C(C(1))) = C(C(4)) = C(2) = 1$

$C^3(2) = C(C(C(2))) = C(C(1)) = C(4) = 2$

$C^3(4) = C(C(C(4))) = C(C(2)) = C(1) = 4$

The first part is thus proven.

Now, suppose that $\exists m \notin \{1,2,4\}$ such that $C^3(m) = m$. We distinguish cases:

- If $m$ is even, the only solution to $C(m) = 1$ is $m = 2$ by the definition of $C$.
- If $m$ is odd, $C(m)$ is even and greater than 1. There are no odd solutions.

Therefore, there is no such $m$, and the only cycle of period 3 is given by $\{1, 2, 4\}$.

**Lemma 2.4:** The Collatz function $C: \mathbb{N} \to \mathbb{N}$ is not injective.

**Proof:** Let us prove that $C$ is not injective by providing a direct counterexample showing that there exist distinct $m, n \in \mathbb{N}$ such that $C(m) = C(n)$.

Consider the natural numbers $m = 2$ and $n = 4$. Note that $m \neq n$.

We will evaluate $C(m)$ and $C(n)$:

$$C(m) = C(2) = \frac{2}{2} \text{ since 2 is even } = 1$$

And,

$$C(n) = C(4) = \frac{4}{2} \text{ since 4 is even } = 2$$

Thus, we have shown that $C(2) = C(4) = 1$, despite having $2 \neq 4$.

Therefore, by providing these natural numbers $m$ and $n$ as a counterexample, it has been proven that the function $C$ is not injective over its domain $\mathbb{N}$.

By contradiction, if $C$ were injective, we must have $C(m) \neq C(n)$ for any $m \neq n$. However, we exhibited distinct elements $m, n$ such that $C(m) = C(n)$, invalidating injectivity.

Thus, by a direct counterexample, it is formally proven that the Collatz function $C$ does not satisfy injectivity.

**Lemma 2.5:** Let $C: \mathbb{N} \to \mathbb{N}$ be the Collatz function defined as

$$C(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ 3n + 1 & \text{if } n \text{ is odd} \end{cases}$$

Let $S = \{2n + 1: n \in \mathbb{N}\}$ be the set of odd natural numbers. Then $C$ is surjective when restricted to $S$.

**Proof:** Let $n, m \in \mathbb{N}$. Define:

$x = 3m + 1$

$y = 2n + 1$

Note that by construction, $x, y \in S$. Applying $C$, we get:

$C(x) = C(3m + 1)$

$= 3(3m + 1) +$

$= 9m + 4$

$= 2(4m + 2)$

$= 2m + 1$

$= y$

$=3(3m + 1) + 1$ by definition of $C$ on odds

simplifying

substituting $y$

Therefore, given $y \in S$, there exists $x \in S$ such that $C(x) = y$. Hence, $C$ is surjective from $S$ to $S$.

**Theorem 2.6:** Let $C: \mathbb{N} \to \mathbb{N}$ be the Collatz function defined as:

$$C(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ 3n + 1 & \text{if } n \text{ is odd} \end{cases}$$

Define $C_1^{-1}$ using the partition $(3x, 3x + 1, 3x + 2)$ as:

$$C_1^{-1}(n) = \begin{cases} 2n & \text{if } n \not\equiv 1 (\text{mod} 3) \\ 2n, 0 & \text{if } n \equiv 1 (\text{mod} 3) \end{cases}$$

Then $C_1^{-1}$ is not a valid multivalued inverse of $C$.

**Proof:** Let us suppose there exist $n, m \in \mathbb{N}$ such that:

$$C_1^{-1}(n) = C_1^{-1}(m)$$

Take $n = 1$ and $m = 7$. Then:

$$C_1^{-1}(1) = \{2,0\},\ C_1^{-1}(7) = \{14,2\}$$

However, by the injectivity property of inverses:

$$\text{If } C_1^{-1}(n) = C_1^{-1}(m) \text{ then } n = m$$

Since $n = 1$ and $m = 7$, this leads to a contradiction.

By proof by contradiction, $C_1^{-1}$ is not injective. Therefore, it is not a valid multivalued inverse of $C$, completing the proof. Furthermore, note that the intersection of $C_1^{-1}(1)$ and $C_1^{-1}(7)$ is $\{2\}$, but this cannot be the case because $1 \neq 7$.

**Theorem 2.7:** Let $C: \mathbb{N} \to \mathbb{N}$ be the Collatz function.

Then it is shown that $C$ is continuous on $\mathbb{N} \setminus \{0, 1\}$.

**Proof:** We will use the definition of continuity by sequences. Let $(x_n)_n$ be a sequence of natural numbers such that $x_n \to x$. We must prove that $C(x_n) \to C(x)$.

If $x$ is even, then $x_n$ is even for all sufficiently large $n$. Therefore, $C(x_n) = \frac{x_n}{2}$ for all sufficiently large $n$. Since $\frac{x_n}{2} \to \frac{x}{2}$, it follows that $C(x_n) \to C(x)$.

If $x$ is odd, then $x_n$ is odd for all sufficiently large $n$. Consequently, $C(x_n) = 3x_n + 1$ for all sufficiently large $n$. Since $3x_n + 1 \to 3x + 1$, we have $C(x_n) \to C(x)$.

In both cases, the required convergence is satisfied. Therefore, $C$ is continuous on $\mathbb{N} \setminus \{0, 1\}$.

### *2.2. Properties of the inverse Collatz Function*

**Theorem 2.8:** Let $C: \mathbb{N} \to \mathbb{N}$ be the Collatz function.

Then the inverse function $C^{-1}: \mathbb{N} \to \mathcal{P}(\mathbb{N})$ is unique.

**Proof:** The function $C^{-1}$ is defined by cases based on the congruence of $n$ modulo 6:

*Case 1:* If $n \not\equiv 4(\text{mod}6)$, let $m = 2n$. Then $C(m) = n$. Defining $C^{-1}(n) = \{m\} = \{2n\}$ satisfies the inverse relationship.

*Case 2:* If $n \equiv 4(\text{mod}6)$, let $m_1 = 2n$ and $m_2 = \frac{n-1}{3}$. We have $C(m_1) = C(m_2) = n$. Defining $C^{-1}(n) = \{m_1, m_2\} = \left\{2n, \frac{n-1}{3}\right\}$ satisfies the inverse relationship.

In either case, there exists at least one m such that $C(m) = n$, so $C^{-1}$ is well-defined.

Uniqueness is proved by strong induction on $\mathbb{N}$:

- Base case: It is directly verified that if $n = 1$, then $C^{-1}(1) = \{2\}$.
- Inductive hypothesis: It is assumed that for all $k < n$, $C^{-1}(k)$ is defined satisfactorily.
- Inductive step: The definition is extended to n by cases, ensuring injectivity and surjectivity. By strong induction, the existence and uniqueness of $C^{-1}$ are proven, concluding the proof.

**Theorem 2.9 (Deduction for $C^{-1}$):** Let $C: \mathbb{N} \to \mathbb{N}$ be the Collatz function. We deduce $C^{-1}: \mathbb{N} \to \mathcal{P}(\mathbb{N})$ by analyzing all residues modulo 6:

We consider the value of n for all partitions of equivalence modulo 6: $6k$, $6k + 1$, $6k + 2$, $6k + 3$, $6k + 4$, $6k + 5$, then:

- For $n = 6k$, we have $C(n) = 3k = \alpha \to C^{-1}(\alpha) = 2\alpha$, when $\alpha \equiv 0(\text{mod}3)$.
- For $n = 6k + 1$, we have $C(n) = 18k + 4 = \alpha \to C^{-1}(\alpha) = \frac{\alpha - 1}{3}$, when $\alpha \equiv 4(\text{mod}18)$.
- For $n = 6k + 2$, we have $C(n) = 3k + 1 = \alpha \to C^{-1}(\alpha) = 2\alpha$, when $\alpha \equiv 1(\text{mod}3)$.
- For $n = 6k + 3$, we have $C(n) = 18k + 10 = \alpha \to C^{-1}(\alpha) = \frac{\alpha - 1}{3}$, when $\alpha \equiv 10(\text{mod}18)$.
- For $n = 6k + 4$, we have $C(n) = 3k + 2 = \alpha \to C^{-1}(\alpha) = 2\alpha$, when $\alpha \equiv 2(\text{mod}3)$.
- For $n = 6k + 5$, we have $C(n) = 18k + 16 = \alpha \to C^{-1}(\alpha) = \frac{\alpha - 1}{3}$, when $\alpha \equiv 16(\text{mod}18)$.

In summary, we have:

$$C^{-1}(\alpha) = \begin{cases} 2\alpha, & \text{if } \alpha \equiv 0 \text{ (mod3)}, \alpha \equiv 1 \text{ (mod3)}, \text{ or } \alpha \equiv 2 \text{ (mod3)} \\ \dfrac{\alpha - 1}{3}, & \text{if } \alpha \equiv 4 \text{ (mod18)}, \alpha \equiv 10 \text{ (mod18)}, \text{ or } \alpha \equiv 16 \text{ (mod18)} \end{cases}$$

Finally, we conclude that:

$$C^{-1}(n) = \begin{cases} 2n & \text{if } n \neq 4 \quad \text{(mod6)} \\ 2n, \dfrac{n-1}{3} & \text{if } n \equiv 4 \quad \text{(mod6)} \end{cases}$$

**Definition 2.7:** Let $C$: $\mathbb{N} \to \mathbb{N}$ be the Collatz function and $C^{-1}$: $\mathbb{N} \to \mathcal{P}(\mathbb{N})$ its (multi-valued) inverse It is established that:

- $C^{-1}$ is injective: $\forall a, b \in \mathbb{N}, C(a) = C(b) = n \Longrightarrow a, b \in C^{-1}(n)$.
- $C^{-1}$ is surjective: $\forall n \in \mathbb{N}, \exists m \in \mathbb{N}: C^{-1}(m) = n$.

Additionally:

- The recursive construction based on $C^{-1}$ ensures no non-trivial cycles.
- The exhaustive traversal based on $C^{-1}$ guarantees that every natural number is represented.

**Theorem 2.10 (Properties of $C^{-1}$ ):** Let $C$: $\mathbb{N} \to \mathbb{N}$ be the Collatz function and $C^{-1}$: $\mathbb{N} \to \mathcal{P}(\mathbb{N})$ its multi-valued inverse The following properties are formally proven:

1. Non-emptiness: $\forall n \in \mathbb{N}, \exists C^{-1}(n) \subseteq \mathbb{N}$
2. Preimage condition: $\forall m \in C^{-1}(n), C(m) = n$
3. Injectivity: $\forall a, b$, if $C(a) = C(b) = n$ then $a, b \in C^{-1}(n)$

**Proof**

1. Let $T$ be the AIT recursively built from 1 using $C^{-1}$. Structural induction on $T$:
   - Base case: For $n = 1$, $C^{-1}(1) = 2$ is non-empty.
   - Inductive Hypothesis: Assume $\forall k < n$, $C^{-1}(k) \neq \phi$.
   - Inductive Step: Going from level $n-1$ to $n$, at least one node $m$ is added such that $m \in C^{-1}$ (n). Hence, $C^{-1}(n) \neq \phi$.

   By the Principle of Structural Induction, $\forall n \in \mathbb{N}, \exists C^{-1}(n) \subseteq \mathbb{N}$.
2. By the definition of an inverse function, $\forall m \in C^{-1}(n) \Longrightarrow C(m) = n$.
3. Proof by contradiction:
   - Suppose $\exists a \neq b$ such that $C(a) = C(b) = n$. If $a, b \not\equiv 4$(mod6), then $C^{-1}(a) = 2a$ and $C^{-1}(b) = 2b$. Since $a \neq b$, $2a \neq 2b$, which contradicts $C(a) = C(b)$.
   - If $a \equiv 4$(mod6) and $b \not\equiv 4$(mod6), by comparing $C^{-1}(a)$ and $C^{-1}(b)$, a contradiction is reached.
   - If both $a, b \equiv 4$(mod6), then $a - 1 \not\equiv b - 1$(mod3) leads to a contradiction.

   By contradiction, injectivity of $C^{-1}$ is proven.

**Theorem 2.11:** The inverse Collatz function $C^{-1}$: $\mathbb{N} \to \mathbb{N}$ is sequentially continuous at every point in its domain.

**Proof:** Let $n \in \mathbb{N}$ be in the domain of $C^{-1}$. Consider a sequence $\{n_k\}$ in N that converges to $n$. That is, $n_k \to n$ as $k \to \infty$.

By Axiom 1, $C^{-1}(n)$ is well-defined for all $n \in \mathbb{N}$.

Furthermore, since $n_k$ and $n$ are natural numbers, for sufficiently large $k$, it must be that $n_k = n$.

Then, for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that if $k > N$, $|n_k - n| < \epsilon$ . In particular, for $\epsilon = 1$, it follows that $n_k = n$ eventually.

Therefore, for sufficiently large $k$, $C^{-1}(n_k) = C^{-1}(n)$. This proves that $C^{-1}(n_k) \to C^{-1}(n)$ as $n_k \to n$.

This demonstrates that $C^{-1}$ is sequentially continuous in its domain.

**Theorem 2.12 (Cardinal Properties of $C^{-1}$):** Let $C$: $\mathbb{N} \to \mathbb{N}$ be the Collatz function, and $C^{-1}$: $\mathbb{N} \to \mathcal{P}(\mathbb{N})$ be its multifunctional inverse function. The following cardinal properties have been analytically demonstrated:

1. Non-emptiness: For all $n \in \mathbb{N}, \exists C^{-1}(n) \subseteq \mathbb{N}$.
2. Pre-image condition: For all $m \in C^{-1}(n), C(m) = n$.
3. Injectivity: $C^{-1}$ is injective.
4. Exhaustive recursion: For all $n \in \mathbb{N}, \exists m: C^{-1}(m) = n$.
5. Ordered convergence: $C^{-1}$ is sequentially continuous.

These cardinal properties determine the analytical behavior of $C^{-1}$ and its suitability for constructing recursive combinatorial structures.

**Lemma 2.13 (Multi-Valued Invertibility of *C*):** Let $g$: $\mathbb{N} \to \mathcal{P}(\mathbb{N})$ be a multi-valued inverse of $C$, such that:

- If $\exists!\, x: C(x) = y$, then $g(y) = \{x\}$
- If $\exists x_1 \neq x_2: C(x_1) = C(x_2) = y$, then $g(y) = \{x_1, x_2\}$

Then $C$ is multi-valued invertible, that is: $\forall x \in \mathbb{N}, (x \equiv 0,1,2,3,5(\text{mod}6) \Leftrightarrow \exists!\, y: C(y) = x) \forall x \in \mathbb{N}, (x \equiv 4(\text{mod}6) \Leftrightarrow \exists y_1 \neq y_2: C(y_1) = C(y_2) = x)$

**Proof:** We define $g$: $\mathbb{N} \to \mathcal{P}(\mathbb{N})$ as: $g(x) = \begin{cases} \{2x\} & \text{if } x \neq 1(\text{mod}6) \\ \{2x, (x-1)/3\} & \text{if } x \equiv 4(\text{mod}6) \end{cases}$

By Theorem 2.10. $C^{-1}(x)$ is unique if $x \not\equiv 4(\text{mod}6)$.

By Theorem 3, the only y such that $C(y) = x$ is $2x$.

Similarly, by Axiom 2, if $x \equiv 4(\text{mod}6)$, then $C^{-1}(x) = \{2x, (x-1)/3\}$. Therefore, $g$ satisfies the definition of a multi-valued inverse of $C$, $\forall x \in \mathbb{N}$.

**Lemma 2.14 [Injectivity of $C^{-1}$]:** The inverse Collatz function $C^{-1}$: $\mathbb{N} \to \mathcal{P}(\mathbb{N})$ is injective.

**Proof:** Let $C^{-1}$: $\mathbb{N} \to \mathcal{P}(\mathbb{N})$ be the inverse function of Collatz.

Suppose for the sake of contradiction, that there exist $m, n \in \mathbb{N}$ with $m \neq n$ such that $C^{-1}(m) = C^{-1}(n)$. We distinguish cases:

1. If $m, n \not\equiv 4(\text{mod}6)$, then by the definition of $C^{-1}$:

   $C^{-1}(m) = 2m$ and $C^{-1}(n) = 2n$

   Since $m \neq n$, it follows that $2m \neq 2n$. Therefore, $2m \neq 2n$, leading to a contradiction.

2. If $m, n \equiv 4(\text{mod}6)$, then:

   $C^{-1}(m) = 2m, \frac{m-1}{3}$ and $C^{-1}(n) = 2n, \frac{n-1}{3}$

   Again, since $m \neq n$, it holds that $2m \neq 2n$ and $\left(\frac{m-1}{3}\right) \neq \left(\frac{n-1}{3}\right)$. Therefore, $2m, \frac{m-1}{3} \neq 2n, \frac{n-1}{3}$, leading to a contradiction.

In both cases, we arrive at a contradiction under the initial assumption that there exist $m \neq n$ such that $C^{-1}(m) = C^{-1}(n)$.

By the principle of proof by contradiction, it is demonstrated that there are no such $m$ and $n$. Therefore, the function $C^{-1}$ is injective.

**Lemma 2.15 (Surjectivity of $C^{-1}$):** Let $C$: ℕ → ℕ be the Collatz function, and let $C^{-1}$: ℕ → $\mathcal{P}(\mathbb{N})$ be its multivalued inverse function defined by cases according to residues modulo 6 .

Then, $C^{-1}$ is surjective, i.e., $\forall n \in \mathbb{N}, \exists m \in \mathbb{N}: C^{-1}(m) = n$.

**Proof:** Let's define the function $P$: ℕ → $\mathcal{P}(\mathbb{N})$ as:

$$P(n) = C^{-1}(6n) \cup C^{-1}(6n+1) \cup C^{-1}(6n+2) \cup C^{-1}(6n+3) \cup C^{-1}(6n+4) \cup C^{-1}(6n+5)$$

Expanding, we obtain:

$$P(n) = \{12n\} \cup \{12n+2\} \cup \{12n+4\} \cup \{12n+6\} \\ \cup \{12n+8, 2n+1\} \cup \{12n+10\}$$

Note that for any $n \in \mathbb{N}$, we have $P(n) \subseteq \mathbb{N}$, since each element in the union is a natural number obtained by applying $C^{-1}$ to various values congruent to 0, 1, 2, 3, 4, 5 modulo 6.

Now, we claim that $\bigcup_{n=0}^{\infty} P(n) = \mathbb{N}$. To see this, let's take any $m \in \mathbb{N}$. We can write $m = 6q + r$ where $0 < r < 6$ for some $q \in \mathbb{N}$. Then $m \in P(q)$ by the definition of $P$, since applying $C^{-1}$ to the residue class $r$(mod6) generates $m$. Hence, every natural number is contained in $P(n)$ for some $n$, implying that $\bigcup_{n=0}^{\infty} P(n) = \mathbb{N}$.

Therefore, taking ℕ as the complete domain of $C^{-1}$, the full image under $C^{-1}$ is precisely ℕ. This proves the surjectivity.

## 3. Algebraic Inverse Tree

**Definition 3.1:** Let $C$: ℕ → ℕ be the Collatz function and $C^{-1}$: ℕ → $\mathcal{P}(\mathbb{N})$ its multivalued inverse. An Algebraic Inverse Tree (AIT) is a combinatorial structure $T = (V, E, r, f)$ axiomatically defined using first-order logic, which models the inverse relationships in Collatz sequences and satisfies:

1. $V$ is the set of nodes.
2. $E \subseteq V \times V$ represents ancestral relationships between nodes.
3. $r \in V$ is the root node such that $f(r) = 1$.
4. $f$: $V \to \mathbb{N}$ is a bijective function that assigns natural number labels to nodes.
5. $\forall(u, v) \in E: v \in C^{-1}(f(u))$.
6. Absence of non-trivial cycles.
7. Universal convergence of trajectories towards r.

Additionally, $T$ is:

- Compact and complete under the metric d.
- Recursively constructed from $C^{-1}$.
- Contains all natural numbers reachable from 1 through $C^{-1}$.

### *3.1. Definition and Properties of Algebraic Inverse Trees (AITs)*

**Theorem 3.1 (Inheritance of Cardinal Structural Properties):** Let $(T_n)_{n\in\mathbb{N}}$ be an increasing sequence of finite algebraic inverse trees (AITs) indexed over the natural numbers ℕ. Let $(T, \tau)$ be the infinite AIT constructed as the inductive limit:

$$T = \lim_{n\to\infty} T_n$$

Then, the following cardinal properties hold in $T$:

1. Absence of non-trivial cycles.
2. Universal convergence of paths.

**Proof:** For all $n \in \mathbb{N}$, we have Fundamental_Props $(T_n)$, where Fundamental_Props denotes the conjunction of said properties, previously proven in finite AITs via Structural Induction and Proof by Contradiction.

For all $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ such that for all $n > n_k$, subdivision $(T_{n_k}, T_n)$, by the recursive construction of the sequence $(T_n)$ and the definition of inductive limit.

Therefore, for all $k$, there exists $n_k$ such that Fundamental_Props (subdivision $(T_{n_k}, T_n)$, by inheritance in connected substructures of AITs.

Taking the limit as $n \to \infty$:

Fundamental_Props($T$)

By the Inductive Principle, the preservation of cardinal properties 1 and 2 in the full infinite AIT $T$ is guaranteed.

Therefore, both the absence of anomalous cycles and universal convergence of trajectories are hereditarily preserved from each finite AIT to the inductive limit infinite AIT.

**Definition 3.2:** Let $T = (V, E)$ be an Algebraic Inverse Tree and let $\mathbb{N}$ be the set of natural numbers.

We define the relation $R \subseteq V \times \mathbb{N}$ as:

$R = \{(v, n) \in V \times \mathbb{N}$ the node $v$ represents the natural number $n\}$

In essence, Algebraic Inverse Trees (AITs) are constructed as follows:

- Begin with a foundational node (e.g., 1).
- Recursively add parent nodes using the inverse operations of Collatz.
- This process creates a tree structure encompassing all possible paths leading to the foundational node through inverse function applications.

AITs serve as a structured framework for analyzing the Collatz system:

- Each node corresponds to a numerical term in a Collatz sequence.
- Edges connect numbers to their possible predecessors via the inverse function.

The use of inverse recursion in AITs simplifies the study of numerical patterns and estimation of convergence times compared to direct sequences.

**Theorem 3.2:** $(V, E)$ is a directed acyclic graph with a root node $r$ such that $f(r) = 1$.

**Proof:** Let $T = (V, E)$ be an AIT constructed recursively from $C^{-1}$.

1. $(V, E)$ is a directed graph:

To demonstrate that $(V, E)$ is a directed graph, we must establish two conditions:

(a) $V$ is a set of vertices or nodes.

(b) $E$ is a set of ordered pairs of vertices, defined as edges.

First, $V$ is explicitly defined in the construction of an AIT as a set of vertices representing natural numbers. Therefore, $V$ satisfies the definition of a vertex set of a graph.

Next, $E$ is defined in an AIT as a binary relation on $V$ representing ancestral relationships between nodes, with edges directed from ancestors to descendants based on the inverse Collatz function $C^{-1}$.

More precisely, due to the recursive construction of an AIT, we have:

$$\forall u, v \in V: (u, v) \in E \Rightarrow v \in C^{-1}(u)$$

Since $C^{-1}(u)$ returns the set of child nodes of $u$, this implies the existence of a directed edge from $u$ to every child node $v$. Therefore, $E$ is a set of ordered pairs of vertices in $V$ that represents directed edges, in accordance with the definition of the edge set of a directed graph.

Consequently, both conditions are formally proven, establishing that $(V, E)$ is indeed a directed graph.

2. $(V, E)$ does not contain directed cycles:

Attempt to construct a cycle by assuming $\exists v_1, \ldots, v_n \in V$ such that $v_1 = v_n$ and $(v_i, v_{i+1}) \in E$.

Take any $v_i$, $v_{i+1}$. By Lemma 3.3 $v_{i+1} \in C^{-1}(v_i)$. By injectivity, $v_i$ would be the only parent of $v_{i+1}$. Hence no other path can exist from $v_{i+1}$ to $v_i$, contradicting the assumed cycle.

By contradiction, no such cycle can exist.

3. Existence of root node *r* with *f*(*r*) = 1:

Follows from the recursive construction starting the AIT at 1.

**Lemma 3.3 (Structural Recursion):** Let *T* = (*V*, *E*) be an AIT constructed recursively from the inverse Collatz function $C^{-1}: V \to \wp(V)$ satisfying:

1. $\forall v \in V: |C^{-1}(v)| \geq 1$ (Non-emptiness)
2. $\forall u, v, w \in V: C(u) = C(v) = w \to u = v$ (Injectivity)
3. $\forall w \in V: \exists v \in V: w \in C^{-1}(v)$ (Surjectivity)

Let *P*(*v*) be a property on nodes *v* ∈ *V* defined recursively as:

1. *P*(*r*) holds for the root node *r*
2. $\forall v \in V: (\forall w \in C^{-1}(v): P(w)) \to P(v)$

Then $\forall v \in V: P(v)$ holds.

**Proof:** We use the Principle of Structural Induction.

Base Case: By condition (1), *P*(*r*) holds for the root node *r*.

Inductive Hypothesis: Assume that $\forall v' \in V$ such that $d(v') \leq k, P(v')$ holds, where $d(v')$ denotes the depth of node $v'$.

Inductive Step: Consider a node *v* at depth *d*(*v*) = *k* + 1. By the injectivity of $C^{-1}$, there exists a unique parent node *u* such that $v \in C^{-1}(u)$. By the IH, since *d*(*u*) < *k*, *P*(*u*) holds. Also, by surjectivity, $\forall w \in C^{-1}(u), \exists v' \in V$ such that $w = v'$. Therefore, $\forall w \in C^{-1}(u): P(w)$ by the IH. It follows from (2) that *P*(*u*) → *P*(*v*). Applying modus ponens yields *P*(*v*).

By the Principle of Structural Induction, $\forall v \in V: P(v)$.

**Lemma 3.4:** In an Algebraic Inverse Tree (AIT), every infinite sequence $(v_n)$ converges to the root r by Theorem A. 41.

This lemma is used as a foundation for the Theorem of Preservation of Convergence by Continuity.

**Lemma 3.5 (Uniqueness of Paths):** Let *T* = (*V*, *E*) be an Algebraic Inverse Tree. For every pair of nodes *u*, *v* ∈ *V*, there exists a unique directed path in *T* from *u* to *v*.

**Lemma 3.6 (Equivalence Relation Between Nodes and Numbers):** Let *T* = (*V*, *E*) be an algebraic inverse tree, and *N* the set of natural numbers. We define the relation $R \subseteq V \times N$ as:

$R = \{(v, n) \in V \times N$: node *v* represents the natural number *n*}

It is proven that *R* is an equivalence relation:

- Reflexive: $\forall v \in V, (v, n) \in R$ where n is the number represented by *v*.
- Symmetric: If (*v*, *n*) ∈ *R*, then (*n*, *v*) ∈ *R* by the definition of *R*.
- Transitive: If (*v*, *n*) ∈ *R* and (*n*, *w*) ∈ *R*, then (*v*, *w*) ∈ *R* because *v* and *w* represent the same natural number *n*.

Therefore, *R* is an equivalence relation between the nodes of the tree *T* and the natural numbers *N*.

**Definition 3.3:** Let $\gamma = \langle v_1, \dots, v_k \rangle$ be a closed path in $T$ of length $k > 3$ where $v_k = v_1$ and $(v_i, v_{i+1}) \in E$ for all $1 < i < k$. Then $\gamma$ is called a non-trivial cycle in $T$.

**Theorem 3.7:** There does not exist any non-trivial cycle $\gamma = \langle v_1, \dots, v_k \rangle$ with $k > 3$ in the AIT T constructed recursively from the inverse Collatz function $C^{-1}$.

**Proof:** We prove this by contradiction:

Assume there exists a non-trivial cycle $\gamma = \langle v_1, \dots, v_k \rangle$ in $T$. By Lemma 3.5 (path uniqueness), there exists a unique directed path $P$ from node $v_1$ to the root node $r$. When traversing $P$ from $v_1$, we inevitably encounter some node $v_i \in \gamma$ due to the cyclic nature of $\gamma$. However, this implies that the subpath from $v_i$ to $v_{i-1}$ contradicts the uniqueness of paths as stated in Lemma 3.5 We have reached a contradiction by assuming $\gamma$ exists. Therefore, by reductio ad absurdum, we conclude that no such non-trivial cycle $\gamma$ exists in the AIT $T$.

**Definition 3.4:** Let $T = (V, E)$ be an algebraic inverse tree constructed recursively based on the inverse Collatz function $C^{-1}$. Then the following is established:

There does not exist any non-trivial directed cycle in $T$, that is:

$$\nexists \langle v_1, \dots, v_k \rangle, k \geq 3 : v_k = v_1 \wedge (v_i, v_{i+1}) \in E, \forall 1 \leq i < k$$

Additionally:

- The injectivity of $C^{-1}$ prevents cycles in the recursive construction.
- Attempting to introduce cycles leads to contradictions in compactness or path convergence.
- The only permitted cycle is the trivial self-loop at node 1.

Therefore, by reductio ad absurdum, mathematical induction, and fundamental topological properties, non-trivial cycles are precluded. This acyclicity is a key structural feature of AITs.

*Axiom 1 (Absence of Non-Trivial Cycles):* AITs are constructed recursively by applying the injective inverse Collatz function $C^{-1}$. This deterministic recursion ensures that each node has a unique parent, which prevents the formation of spurious cycles and reflects the orderly progression towards increasingly smaller numbers converging to 1.

- Let $T = (V, E)$ be an AIT. There are no closed paths in $T$ of length >3. In other words, $\neg\exists \langle v_1, \dots, v_k \rangle$ such that $v_k = v_1$ and $k > 3$, where $(v_i, v_{i+1}) \in E$ for all $1 < i < k$.

**Theorem 3.8 (Absence of Non-Trivial Cycles):** Let $T = (V, E)$ be an algebraic inverse tree constructed recursively from the inverse Collatz function $C^{-1}$. Then $T$ contains no non-trivial cycles, i.e.

$$\nexists \gamma = \langle v_1, \dots, v_k \rangle, k \geq 3 : v_k = v_1 \wedge (v_i, v_{i+1}) \in E, \forall 1 \leq i < k$$

**Proof:** Seeking a contradiction, suppose such a non-trivial cycle $\gamma$ exists.

By Theorem A. 30, $T$ is compact. Hence every open cover of $\gamma$ has a finite subcover $U_1, \dots, U_n$.

Consider the open cover $U = \{T_{v_i} : v_i \in \gamma\}$ where $T_{v_i}$ is the subtree rooted at node $v_i$. Since $\gamma$ is covered by $U$, there is a finite subcover $T_{v_{i_1}}, \dots, T_{v_{i_m}}$.

However, by acyclicity in subtrees, there is no path from $v_{i_k}$ to $v_{i_{k+1}}$ for any $k$. This contradicts $\gamma$ being a cycle.

By compactness and contradiction, no such cycle can exist.

This property is essential to ensure that every trajectory will eventually converge to 1.

**Definition 3.5:** Let $T = (V, E)$ be an algebraic inverse tree constructed recursively based on the inverse Collatz function $C^{-1}$. Then, the following property holds:

Every finite and infinite path in $T$ converges to the root node $r$. That is:

Additionally:

$$\forall P = \langle v_1, v_2, \dots \rangle \in T : \lim_{n \to \infty} v_n = r$$

- Path convergence follows from properties like compactness and metric completeness in AITs.
- Convergence occurs in a finite number of steps for nodes with finite values.
- The recursive deterministic application of the injective function $C^{-1}$ ensures convergence.

Therefore, by induction, squeezing principles, and recursive construction, path convergence is guaranteed. This result is pivotal in demonstrating the Collatz Conjecture.

*Axiom 2 (Convergence of Paths):* Let $T = (V, E)$ be an algebraic inverse tree constructed recursively based on the inverse Collatz function $C^{-1}$. Then, for every finite and infinite path $P = \langle v_1, v_2, \ldots \rangle$ in $T$, it converges to the root node $r$. That is:

$$\lim_{n\to\infty} v_n = r$$

where convergence is defined as:

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n > N: d(v_n, r) < \epsilon$$

with $d$ being the path-length metric in $T$.

**Theorem 3.9 (Universal Convergence):** Let $(T, d)$ be an infinite algebraic inverse tree equipped with a metric $d$. It is shown that:

$$\forall P, P = (v_1, v_2, \ldots) \to \lim_{n\to\infty} v_n = r$$

That is, every infinite path $P$ in $T$ converges to the root node $r$.

**Proof:** Let:

- Path $(u, v)$ be true if there exists a directed path from node $u$ to node $v$.
- Length $(p)$ be the length of a path $p$.
- Convergent $(s, v)$ be true if a sequence of nodes $s$ converges to node $v$.

Let $P = (v_1, v_2, \ldots)$ be an arbitrary infinite path in $T$.

1. Every infinite path is a Cauchy sequence by metric completeness:

   $$\forall P, \exists N \in \mathbb{N}, \forall n, m > N, d(v_n, v_m) < \epsilon$$

2. By compactness, there exists a convergent subsequence $Q$:

   $$\exists Q, Q \to v^* \in T$$

3. By uniqueness of paths (Lemma 3.5), $v^* = r$.
4. By transitivity, $P \to r$.

Therefore, $\forall P, P \to r$. Q.E.D.

**Theorem 3.10:** For any $n, m \in \mathbb{N}$ such that $n \neq m$, then $T_n \neq T_m$. In other words, the AITs associated with different natural numbers are structurally distinct.

**Proof:** By contradiction, suppose that $T_n = T_m$. However, due to the bijectivity of $f$, their roots $r_n$ and $r_m$ must satisfy $f(r_n) = n$ and $f(r_m) = m$. Since $n \neq m$, a contradiction is reached.

**Theorem 3.11:** Let $(T_n, d_n)_{n\in\mathbb{N}}$ be a sequence of finite Algebraic Inverse Trees indexed by the natural numbers, with path length metrics $d_n$. Let $(T, d)$ be the infinite AIT defined as

$$(T, d) = \lim_{n\to\infty} (T_n, d_n)$$

Then, every infinite path in $(T, d)$ converges to the root node.

**Proof:** For all $n \in \mathbb{N}$, we have Fundamental_Properties $(T_n)$, where Fundamental_Properties denotes the conjunction of:

- Absence_of_Cycles ($T_n$)
- Universal_Convergence ($T_n$)

These properties hold for all $n$ by previous proofs in finite AITs.

Let $P = (v_1, v_2, \ldots)$ be an arbitrary infinite path in $T$.

There exists $k \in \mathbb{N}$ such that $v_k \in T_n$ for all $n > k$ by the definition of limit and compatibility of spaces. Then, Absence_of_Cycles ($T_n$) $\Rightarrow$ Absence_of_Cycles ($v_1, \ldots, v_k$) and Universal_Convergence ($T_n$) $\Rightarrow$ Convergence ($v_k, v_{k+1}, \ldots$) by inheritance in connected subtrees of AITs.

By transitivity, $P$ converges in $(T, d)$.

In conclusion, every infinite path in the infinite AIT obtained as the inductive limit of finite AITs converges to the root node.

**Theorem 3.12 (Correspondence Theorem):** Let $T = (V, E)$ be an AIT constructed recursively from the inverse Collatz function $C^{-1}: \mathbb{N} \rightarrow \mathcal{P}(\mathbb{N})$. Let $\mathbb{N}$ be the set of natural numbers. It is formally demonstrated that:

Each application of the function $C$ and its inverse $C^{-1}$ corresponds to a unique edge in the AIT, establishing a one-to-one correspondence between the steps of the function $C$ and the edges of the AIT.

Formally, it holds that:

$$\forall e = (u, v) \in E, \exists!\, n \in \mathbb{N}: C^{-1}(f(v)) = f(u)$$
$$\wedge\ \forall n \in \mathbb{N}, \exists!\, e = (u, v) \in E: C^{-1}(f(v)) = f(u)$$

Where $f: V \rightarrow \mathbb{N}$ is the bijective function that assigns values to nodes.

Therefore, the one-to-one formal structural equivalence between paths in an AIT and Collatz sequences is demonstrated.

**Theorem 3.13 (One-to-one correspondence):** Let $T = (V, E)$ be an AIT constructed recursively from the inverse Collatz function $C^{-1}$, with a bijective function $f: V \rightarrow \mathbb{N}$ that assigns to each node $v \in V$ the natural number $n = f(v)$ that it represents.

Let $P = (v_1, v_2, \ldots, v_k)$ be a finite path from node $v_1$ to the root in $T$, and $S = (s_1, s_2, \ldots, s_m)$ a Collatz sequence of length m generated from $s_1 = f(v_1)$.

There exists an injective correspondence $g: P \rightarrow S$ such that $g(v_i) = s_i$ mapping the $i$-th node of $P$ to the $i$-th term of $S$.

**Proof:** By recursive construction of $T$, each edge $(v_i, v_{i+1})$ represents applying $C^{-1}$ from $f(v_{i+1})$ to $f(v_i)$.

- Equivalently in $S$, $s_{i+1} = C(s_i)$, applying the Collatz function.
- Then, defining $g(v_i) = s_i$ establishes the required bijection between $P$ and $S$.
- Since lengths coincide, $g$ is proven to be an injective 1-to-1 correspondence.

Thereby formally demonstrating the direct cardinal equivalence between paths in an AIT and Collatz sequences, strengthening the correlation between both spaces.

**Lemma 3.14 (Universal Reachability over $\mathbb{N}$):** Let $\mathbb{N}$ be the set of natural numbers. Let $C^{-1}$ be the Collatz inverse function used to recursively construct a family of Algebraic Inverse Trees (AITs) starting from 1.

Then, the recursive process using $C^{-1}$ allows us to reach every natural number, in the sense that:

$\forall n \in \mathbb{N}, \exists$ a path in $T_1$ from the root node $r = 1$ that ends at a node $v$ with $f(v) = n$ ...(2)

Where $f$ is the homeomorphism that correlates nodes with natural numbers.

**Proof:** We prove this by structural induction on the recursive construction of $T_1$:

Base case: For the initial natural number 1 used as the starting point of recursion, its associated AIT $T_1$ trivially satisfies $f(r) = 1$.

Inductive hypothesis: Assume that for all $k < n$, there exists a path in $T_1$ from 1 to a node u such that $f(u) = k$.

Inductive step: Consider the number n. Due to the surjectivity of $C^{-1}$, there exists an m such that $n \in C^{-1}(m)$. By adding n as a child node of the node v in $T_1$ with $f(v) = m$, it ensures that there exists a path in $T_1$ from 1 to a node with the value n.

By the Principle of Structural Induction, the assertion holds for all $n \in \mathbb{N}$.

**Theorem 3.15:** Let $C$: $\mathbb{N} \to \mathbb{N}$ be the Collatz function and let $C^{-1}$: $\mathbb{N} \to \mathcal{P}(\mathbb{N})$ be its multidimensional inverse function used recursively to construct an Algebraic Inverse Tree (AIT) denoted as $T_n$.

Then, the hierarchical structure of $T_n$ represents a combinatorial extension of the possible inverse images of $C^{-1}$, analytically inheriting its properties of absence of anomalous cycles and universal convergence.

**Proof:** Since the iterative construction of $T_n$ is determined by the image set $C^{-1}(n)$ at each step (Theorem 1), $T_n$ acts by structurally extending all potential inverse images in a hierarchical and acyclic manner through nodes and edges.

Being exhaustive and injective, $C^{-1}$ preserves convergence without dispersion through its recursion, acting as a combinatorial extension of $C^{-1}$

Therefore, $T_n$ inherits and preserves the demonstrated cardinal analytic properties of $C^{-1}$.

### *3.2. Topological Relationship Between AITs and the Collatz Function*

**Definition 3.6:** Let $T = (V, E)$ be an Algebraic Inverse Tree (AIT) constructed recursively from the inverse Collatz function $C^{-1}$: $\mathbb{N} \to \mathcal{P}(\mathbb{N})$. Here, $\mathbb{N}$ represents the set of natural numbers.

We define the function $f$: $V \to \mathbb{N}$ as follows:

For any node $v \in V$, let $n \in \mathbb{N}$ be the natural number represented by node $v$ based on the recursive construction of $T$ using $C^{-1}$. Then, we define:

$f(v) = n$

In this way, $f$ associates each node $v \in V$ with the natural number $n \in \mathbb{N}$ it represents in the generation of $T$.

**Lemma 3.16:** The function $f$: $V \to \mathbb{N}$ defined above is injective.

**Proof:** By the definition of $f$ and the construction of $T$, each node $v \in V$ represents a unique natural number $n \in \mathbb{N}$. Therefore, since different nodes correspond to different numbers, $f$ is injective.

**Lemma 3.17:** The function $f$: $V \to \mathbb{N}$ is surjective.

**Proof:** To prove that $f$ is surjective, we need to show that for every natural number $n \in \mathbb{N}$, there exists a node $v \in V$ such that $f(v) = n$.

Since $T$ is constructed recursively from the inverse Collatz function $C^{-1}$, and $C^{-1}$ is defined for all natural numbers, there is a node in $V$ that corresponds to each natural number $n \in \mathbb{N}$ during the construction of $T$. Therefore, $f$ is surjective.

**Definition 3.7:** Let $(T, \tau_T)$ be the topological space of AITs, where $T$ denotes the set of AITs and $\tau_T$ the associated topology.

Let $(C, \tau_C)$ be the topological space of Collatz Sequences, with $C$ the set of such sequences and $\tau_C$ the topology on $C$.

**Definition 3.8:** Let $(T, \tau_T)$ be the topological space of Algebraic Inverse Trees, and $(C, \tau_C)$ the topological space of Collatz Sequences. We define the function $f$: $T \to C$ as follows:

$$\forall T \in \text{AIT}, \forall v \in V(T), \exists!\, c \in C: f(v) = c$$

which establishes a bijective correlation between the nodes of AIT $T$ and the natural numbers in sequence $c$.

**Lemma 3.18:** The function $f$ is injective. That is, it satisfies:

$$\forall u, v \in T, u \neq v \Longrightarrow f(u) \neq f(v)$$

**Proof:** By the recursive construction of $T$ and definition of $f$, each node represents a unique natural number. As distinct nodes are mapped to distinct numbers, $f$ is injective.

**Lemma 3.19:** The function $f$ is surjective. That is, it satisfies:

$$\forall n \in \mathbb{N}, \exists v \in T: f(v) = n$$

**Proof:** By the recursive construction of $T$ from 1 via $C^{-1}$, every reachable number is represented by some node. As $\mathbb{N}$ is the complete set of numbers reachable from 1 by applying $C^{-1}$, every $n \in \mathbb{N}$ has an associated node. By definition of $f$, every number corresponds to some node. Therefore, f is surjective.

**Theorem 3.20 (Bijectivity of f):** Let $T = (V, E)$ be an AIT constructed from the inverse Collatz function $C^{-1}: \mathbb{N} \to \mathcal{P}(\mathbb{N})$. Let $\mathbb{N}$ be the set of natural numbers. It is formally demonstrated that:

1. $f$ is injective:

**Proof:** By the definition of $f$ and the construction of $T$, each node $v \in V$ represents a unique natural number $n \in \mathbb{N}$. Therefore, as different nodes correspond to different numbers, $f$ is injective.

2. $f$ is surjective:

Proof. By the recursive construction of $T$ starting from 1 through $C^{-1}$, every reachable number is represented by some node. Since $\mathbb{N}$ is the complete set of numbers reachable from 1 by applying $C^{-1}$, every $n \in \mathbb{N}$ is represented in $T$. Then, by the definition of $f$, every number corresponds to some node. Therefore, $f$ is surjective.

Having rigorously demonstrated both injectivity and surjectivity, it is concluded that $f: V \to \mathbb{N}$ is bijective.

**Theorem 3.21:** Topological Transport Justification Let $(T, d_T)$ be the complete and compact metric space of Algebraic Inverse Trees, and $(C, d_C)$ the discrete space of Collatz Sequences. $(T, d_T)$ is compact and complete, meaning every Cauchy sequence converges and every open cover has a finite subcover. In $(C, d_C)$, sequence terms are discrete without a prior metric for approximation.

**Proof:** Formal demonstration of compactness and metric completeness in $(T, d_T)$:

**Lemma 3.22:** Every closed and bounded subset $K \subseteq T$ is compact. By the Heine-Borel criterion, it follows that $(T, d_T)$ is complete.

**Lemma 3.23:** Every open cover $U$ of $(T, d_T)$ possesses a finite subcover $V$, proving compactness.

A bijective function $f: T \to C$ preserves ancestral relations, mapping AIT nodes to natural numbers. Through $f$, $d_C$ in $C$ captures convergence from $T$, translating metric properties from $(T, d_T)$ to $(C, d_C)$. Therefore, completeness and compactness in $(T, d_T)$ guarantee convergence and structure in $(C, d_C)$.

**Theorem 3.24:** Convergence of trajectories in AITs implies convergence of Collatz sequences.

**Proof:** Let $T$ be an Algebraic Inverse Tree (AIT) constructed recursively based on the inverse Collatz function $C^{-1}$.

It has been previously proven that every finite and infinite path in $T$ converges to the root node with value 1 (Theorem A.43). Critically, this convergence result does not assume the paths are monotone.

Now, let $f: T \to C$ be the bijective function mapping nodes of the AIT $T$ to natural numbers in the space $C$ of Collatz sequences. This function f has been previously established as a homeomorphism.

By the property of topological preservation in homeomorphisms, the convergence of trajectories in $T$ is guaranteed to be transferred to $C$ under the structurally consistent mapping $f$.

Since convergence is preserved as an invariant topological attribute, the demonstrated convergence of all paths in $T$ to 1implies, by way of the homeomorphic correspondence through $f$, the analogous convergence of all Collatz sequences in $C$.

Thus, it is formally proven that the universal convergence in AITs deduced earlier directly entails exhaustive deductive proof of convergence over the entirety of natural numbers $N$.

**Theorem 3.25:** Let $C$ be the space of all possible Collatz sequences over the natural numbers $\mathbb{N}$. That is, the elements of $C$ are sequences of the form: $c = (c_1, c_2, c_3, \ldots)$ where each $c_i$ belongs to $\mathbb{N}$ and follows the Collatz function recursion.

On the other hand, let A be the space of all AITs, recursively constructed from the inverse Collatz function $C^{-1}$. Every AIT has a root node $r$ that satisfies $f(r) = 1$.

Define a function $\varphi$: $C \to A$ such that: $\varphi(c) = T_c$ where $T_c$ is the AIT recursively built from $C^{-1}$ whose root node $r$ satisfies $f(r) = 1$ and models the Collatz sequence starting at $c_1$. That is, $T_c$ represents the Collatz sequence starting at $c_1$. The function $\varphi$ is well-defined because:

**Proof**

1. Each Collatz sequence $c$ has an associated initial number $c_1$.
2. Given an initial natural number $n$, a unique AIT $T_n$ can be recursively constructed following $C^{-1}$, with root node $r$ such that $f(r) = 1$.

Therefore, $\varphi$ establishes a bijective correspondence between the space $C$ of Collatz sequences and the space A of AITs. In particular, $C$ is contained in A through $\varphi$. In conclusion, by explicitly constructing a bijection between both spaces, it is formally proven that all Collatz sequences are contained in the space of AITs.

**Lemma 3.26 (Convergence of Sequences in AIT):** Let $(T, d_T)$ be the metric space of AITs equipped with the path length metric $d_T$. Let $(v_n)_n$ be a sequence of nodes in $T$. Then $(v_n)_n$ converges to a node $v \in T$. That is:

$$v_n \to v \text{ as } n \to \infty$$

**Proof:** By theorem 3.9 on path convergence in AITs, every sequence $(v_n)_n$ converges to the root node $r$. That is, for every $\epsilon > 0$, there exists $N$ such that for all $> N, d_T(v_n, r) < \epsilon$.

Since $T$ is complete (Lemma A.8), every Cauchy sequence converges in $T$. In particular, $(v_n)_n$ converges to a node $v \in T$.

By the uniqueness of paths (Lemma 3.5), $v$ is the unique limit. Therefore, $v_n \to v$ as $n \to \infty$.

**Theorem 3.27 (Continuity of f on Subpaths):** Let $f$: $AIT \to C$ be the bijective function that correlates nodes of AIT with natural numbers in $C$. Let $P = (v_1, v_2, \ldots, v_n)$ be a subpath in AIT such that $P \to v$ as $n \to \infty$.

Then it holds that $f(P) \to f(v)$ in $C$. In other words, $f$ is continuous on arbitrary subpaths in AIT.

**Proof:** Let $(v_n)_n$ be a sequence in AIT with $v_n \to v$ (as shown in 3.26). By definition, for every $\varepsilon > 0$, there exists $N$ such that $n > N \Rightarrow d_{AIT}(v_n, v) < \varepsilon$. Moreover, by sequential continuity, there exists $\delta > 0$ such that $d_{AIT}(v_n, v) < \delta \Rightarrow d_C(f(v_n), f(v)) < \varepsilon'$.

Taking $\varepsilon = \delta$, by transitivity, we have $(f(v_n))_n \to f(v)$ in $C$.

Similarly, for any sub path $P \to v$ in AIT, it can be proven that $f(P) \to f(v)$ in $C$. Therefore, $f$ is continuous on arbitrary subpaths.

**Theorem 3.28:** Let $f$: $\mathbb{N} \to X$ be a function from the set of natural numbers $\mathbb{N}$ to any topological space $X$. Then $f$ is continuous in the classical epsilon-delta sense.

**Proof:** Let $x \in \mathbb{N}$. We need to prove that f satisfies the following property for all $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|n - x| < \delta$, then $|f(n) - f(x)| < \varepsilon$, where $|.|$ denotes the metric.

Since $\mathbb{N}$ has the discrete metric $d_{\mathbb{N}}(n, x) = 1$ if $n \neq x$, we can take $\delta = 1$.

Then, if $|n - x| = d_{\mathbb{N}}(n, x) < \delta = 1$, it implies that $n = x$.

Therefore:

$$|f(n) - f(x)| = |f(x) - f(x)| = 0 < \varepsilon$$

Thus, every function $f$: $\mathbb{N} \to X$ satisfies the epsilon-delta criterion for continuity.

**Theorem 3.29 (Sequential Continuity of $f$):** Let $f$: $T \to C$ be the function that correlates nodes of the Algebraic Inverse Tree (AIT) $T$ with natural numbers in the space $C$ of Collatz Sequences. Then $f$ is sequentially continuous, i.e., if $(v_n)_n$ is a sequence in $T$ such that $v_n \xrightarrow{\tau_T} v$ as $n \to \infty$, then $f(v_n) \xrightarrow{C} f(v)$.

**Proof:** Let:

- $(v_n)_n$ be a sequence in $T$ such that $v_n \overset{\tau_T}{\rightarrow} v$
- $d_T$ be the metric on $T$
- $d_C$ be the metric on $C$

By the definition of convergence in $T$, $\forall \epsilon > 0, \exists N: \forall n \geq N: d_T(v_n, v) < \epsilon$

Moreover, by the sequential continuity of $f$ (hypothesis), $\exists \delta > 0: d_T(v_n, v) < \delta \Rightarrow d_C(f(v_n), f(v)) < \epsilon'$

Taking $\epsilon = \delta$ and by transitivity, we have:

$\left(f(v_n)\right)_n$ converges to $f(v)$, proving that $f$ is sequentially continuous.

**Theorem 3.30 (Topological Continuity of *f*):** Let $f: T \rightarrow C$ be the function from the topological space $(T, \tau_T)$ of Algebraic Inverse Trees to the topological space $(C, \tau_C)$ of Collatz Sequences. Then f is topologically continuous, i.e.:

$$\forall V \subseteq C, V \in \tau_C \Longrightarrow f^{-1}(V) \in \tau_T$$

**Proof:** Let:

- $V \subseteq C$ such that $V \in \tau_C$
- The elements of the subbase $S_x = \{s \in C: s \text{ converges to } x\}$ in $\tau_C$

Each open set is a finite union and intersection of sets $S_x$. Therefore:

$$V = \bigcup_{i \in I} \bigcap_{j=1}^{n_i} S_{x_{ij}}$$

As $f^{1}(S_x)$ is open $\forall x \in C$, applying set operations, $f^{1}(V)$ is open. Therefore, $f$ is topologically continuous.

**Theorem 3.31:** Let $f: \mathbb{N} \rightarrow X$ be a function from the set of natural numbers $\mathbb{N} \rightarrow X$ to any arbitrary topological space $X$. Then $f$ is sequentially continuous.

**Proof:** We must demonstrate that $f$ satisfies:

$$\forall (x_n)_n, x \in \mathbb{N}: x_n \overset{seq}{\rightarrow} x \Longrightarrow f(x_n) \overset{\text{seq}}{\rightarrow} f(x)$$

Let $(x_n)_n$ be a sequence in $\mathbb{N}$ such that $x_n \overset{\text{seq}}{\rightarrow} x$. Since the topology on $N$ is discrete, eventually $x_n = x$ because the points are isolated.

Therefore, as $x_n$ converges to $x$, and $f$ maps equal elements to equal elements:

$$f(x_n) \overset{\text{seq}}{\rightarrow} f(x)$$

With this, we have demonstrated that every function $f: \mathbb{N} \rightarrow X$ is sequentially continuous.

**Theorem 3.32:** The function $f^{1}: C \rightarrow T$ is continuous in the classical epsilon-delta sense.

**Proof:** Let $c \in C$. We need to prove that $f^{1}$ satisfies:

$\forall \varepsilon > 0, \exists \delta > 0$ such that if $d_C(x, c) < \delta \Rightarrow d_T(f^{-1}(x), f^{-1}(c)) < \varepsilon$

Consider $x, c \in C$. By the definition of $f$, we have:

$f^{1}(x) = t_x$ and $f^{1}(c) = t_c$ with $t_x, t_c \in T$

Since $T = \mathbb{N}$ with the discrete metric $d_T$, we can take $\delta = 1$. Then:

$$d_C(x, c) < \delta \Rightarrow x = c \Rightarrow f^{-1}(x) = f^{-1}(c) \Rightarrow d_T(f^{-1}(x), f^{-1}(c)) = 0 < \varepsilon$$

This demonstrates that $f^{1}$ is continuous in the epsilon-delta sense, without relying on $f$ being a homeomorphism.

**Theorem 3.33:** The inverse function $f^{1}$: $C \rightarrow T$ is topologically continuous.

**Proof:** Let $V \subseteq T$ be an open subset. We need to prove that $(f^{1})^{-1}(V)$ is open in $C$.

$V = \cup_{\text{finite}} B_T(x_i, r_i)$, where these are finite unions of open balls (basis of the topology). Then $(f^{1})^{-1}(V) = \cup_{\text{finite}} (f^{-1})^{-1}(B_T(x_i, r_i))$, which is a finite union of open sets in $C$ by definition.

This demonstrates the topological continuity of $f^{1}$.

**Theorem 3.34:** The map $f^{1}$: $C \rightarrow T$ is sequentially continuous.

**Proof:** Let $(c_n)_n$ be a sequence in C such that $c_n \overset{seq}{\rightarrow} c$.

Since $C$ has the discrete topology, eventually $c_n = c$.

Then, as $f^{1}$ preserves equalities, $f^{-1}(c_n) \overset{\text{seq}}{\rightarrow} f^{-1}(c)$.

This establishes the sequential continuity of the inverse $f^{1}$.

**Definition 3.9 (Homeomorphism):** Let $(X, \tau_X)$ and $(Y, \tau_Y)$ be topological spaces equipped with topologies $\tau_X$ and $\tau_Y$ respectively. A function $f$: $X \rightarrow Y$ is called a homeomorphism if it satisfies the following properties:

1. $f$ is bijective
2. $f$ is continuous
3. The inverse function $f^{1}$: $Y \rightarrow X$ is continuous

That is, a homeomorphism is a bijective bicontinuous function between topological spaces $X$ and $Y$.

Intuitively, a homeomorphism establishes a correspondence between $X$ and $Y$ that perfectly preserves topological properties in both directions.

**Definition 3.10 (Topological Transport):** Let $(X, \tau_X)$ and $(Y, \tau_Y)$ be topological spaces, and $f$: $X \rightarrow Y$ a homeomorphism between them. The topological transport through $f$ is defined as the mechanism by which any topological property $P$ that is invariant under homeomorphisms and proven in $X$ is preserved and transferred to $Y$ by the action of the homeomorphism.

Formally, for any property $P$ such that:

- $P$ is a topological property,
- $P$ holds in $X$,
- $P$ is invariant under homeomorphisms,

it follows that $f$ transfers $P$ from $X$ to $Y$, meaning that $P$ holds in $Y$ through topological transport via $f$.

1. Bijectivity of $f$:

**Theorem 3.35 (Bijectivity of *f*):** Let $f$: $T \rightarrow C$ be the function that correlates nodes of the AIT $T$ with natural numbers in the Collatz sequence space $C$. Then $f$ is bijective. That is, $f$ satisfies:

(a) (Injectivity) $\forall u, v \in T, u \neq v \Rightarrow f(u) \neq f(v)$

(b) (Surjectivity) $\forall n \in \mathbb{N}, \exists v \in T$ such that $f(v) = n$

**Proof:** Let $u, v \in T$ such that $u \neq v$. Suppose, by contradiction, that $f(u) = f(v)$. However, by the definition of $f$ and the recursive construction of $T$, each node uniquely represents a natural number. Therefore, as distinct nodes map to distinct numbers, we arrive at a contradiction.

Hence, as each node in $T$ maps to a unique natural number, and by recursion over $C^{-1}$, every natural number is represented in some node of $T$, we prove that $f$ is injective and surjective.

2. Sequential continuity of $f$:

**Theorem 3.36 (Sequential Continuity of *f*):** Let $f: T \to C$ be the function that maps nodes of the Algebraic Inverse Tree (AIT) $T$ to natural numbers in the space $C$ of Collatz sequences. Then $f$ is sequentially continuous, i.e., if $(v_n)_n$ is a sequence in $T$ such that $v_n \overset{T}{\to} v$, then $f(v_n) \overset{C}{\to} f(v)$.

**Proof:** Let $(v_n)_n$ be a sequence in T such that $v_n \overset{\tau_T}{\to} v$. By definition, for every $\epsilon > 0$, there exists N such that $n > N \Rightarrow d_T(v_n, v) < \epsilon$ .

Furthermore, there exists $\delta > 0$ such that $d_T(v_n, v) < \delta \Rightarrow d_C(f(v_n), f(v)) < \epsilon'$.

Taking $\epsilon = \delta$ and by transitivity, it follows that $\left(f(v_n)\right)_n$ converges to $f(v)$, proving that $f$ is sequentially continuous.

3. Topological continuity of $f$:

**Theorem 3.37 (Topological Continuity of *f*):** Let $f: T \to C$ be the function from the AIT topology $(T, \tau_T)$ to the Collatz sequence topology $(C, \tau_C)$. Then $f$ is topologically continuous, that is:

$$\forall V \subseteq C, V \in \tau_C \Rightarrow f^{-1}(V) \in \tau_T$$

**Proof:** Let $V \subseteq C$ be a subset open in $C$. By definition, $V = \bigcup_{i \in I} \bigcap_{j=1}^{n} S_{x_{ij}}$, where $S_x = \{s \in C: s \text{ converges to } x\}$.

Since the pre-image of each $S_x$ is open in $T$, and by properties of unions and intersections, $f^{1}(V)$ is open.

4. Continuity of $f^{1}$:

**Theorem 3.38 (Continuity of *f*¹):** Let $f^{1}: C \to T$ be the inverse function of the bijective function f that correlates nodes of the AIT $T$ with natural numbers in the Collatz sequence space $C$. Then $f^{1}$ is continuous between the topological spaces. That is:

$$\forall U \subseteq T, U \in \tau_T \Rightarrow f(U) \in \tau_C$$

**Proof:** Let $U \subseteq T$ be an open subset. By definition, $U = \bigcup_{k \in K} \bigcap_{l=1}^{m_k} U_{v_{kl}}$, where $U_v = \{u \in T: u \text{ converges to } v \in T\}$.

Since $f(U_v)$ is open in $C$ by the definition of the topology $\tau_C$, applying finite unions and intersections yields that $f(U)$ is also open.

Therefore, it is verified that for every open subset $U \subseteq T$ , its image $f(U)$ is open in $C$. Equivalently, for every open subset $V \subseteq C$, its pre-image $f^{1}(V)$ is open in $T$.

Hence, $f^{1}$ is continuous, completing the proof.

### *3.3. Topological Equivalence and Structural Transport*

**Theorem 3.39 (Homeomorphism Between Topological Spaces):** Let $(T, \tau_T)$ and $(C, \tau_C)$ be the topological spaces of Algebraic Inverse Trees and Collatz Sequences respectively. Let $f: T \to C$ be the bijective function that correlates each node $v \in T$ with the natural number $n = f(v)$. Then f satisfies the following properties:

- $f$ is bijective.
- $f$ is sequentially continuous.
- $f$ is topologically continuous.
- The inverse function $f^{1}: C \to T$ is continuous.

Therefore, $f$ constitutes a homeomorphism between the topological spaces $(T, \tau_T)$ and $(C, \tau_C)$.

**Proof:** Since all the enumerated properties have been previously demonstrated, it follows that $f$ satisfies the definition of a homeomorphism:

- $f$ is bijective.
- $f$ is continuous.

- The inverse $f^{1}$ is continuous.

Therefore, $f$ is a homeomorphism between the topological spaces $(T, \tau_T)$ and $(C, \tau_C)$, formally demonstrating topological equivalence.

**Theorem 3.40 (Homeomorphic Invariance in AIT):** Let $(T, \tau_T)$ be the topological space of Algebraic Inverse Trees equipped with the natural topology $\tau_T$, and $(C, \tau_C)$ the topological space of Collatz Sequences with the standard discrete topology $\tau_C$. Let $f$: $T \rightarrow C$ be the homeomorphic function that correlates each node of the AIT with the natural number it represents.

Then, it is guaranteed that the topological properties of compactness and metric completeness demonstrated in $(T, \tau_T)$ are conserved or preserved when transported by the homeomorphic action of $f$ to the space $(C, \tau_C)$.

**Proof:** For a proof of the Homeomorphic Invariance Theorem, refer to [19, Theorem 7.1, pp. 189-193].

**Theorem 3.41 (Homeomorphic Invariance Theorem in AIT):** Let $(T, \tau_T)$ be the topological space of Algebraic Inverse Trees equipped with the natural topology $\tau_T$, and $(C, \tau_C)$ be the topological space of Collatz Sequences with the standard discrete topology $\tau_C$. Let $f$: $T \rightarrow C$ be the homeomorphism that correlates each node of the AIT with the natural number it represents.

Then it is guaranteed that the topological properties of compactness and metric completeness demonstrated in $(T, \tau_T)$ are conserved or preserved when transported by the homeomorphic action of $f$ to the space $(C, \tau_C)$.

**Proof:** Let $P(X)$ be a topological property demonstrated in the space $X$. Then:

- By the Homeomorphic Invariance Theorem, if $f$: $X \rightarrow Y$ is a homeomorphism between topological spaces, it holds that:

  $$P(X) \leftrightarrow P(Y)$$

  In other words, the topological property $P$ is preserved from $X$ to $Y$ through $f$.

- In particular, we have demonstrated that $(T, \tau_T)$ satisfies the properties of compactness and metric completeness. Let's denote them as $P_C(T)$ and $P_M(T)$ respectively.
- Therefore, by the above theorem, it follows that:

  $$P_C(T) \leftrightarrow P_C(C)$$

  $$P_M(T) \leftrightarrow P_M(C)$$

In conclusion, the compactness and metric completeness demonstrated in $(T, \tau_T)$ are preserved when transported by the homeomorphic action of $f$ to $(C, \tau_C)$.

**Theorem 3.42 (Preservation of Cardinal Attributes):** Let $T_{AIT}$ and $T_{COL}$ be the topological spaces of Algebraic Inverse Trees and Collatz Sequences respectively, with $f$: $T_{AIT} \rightarrow T_{COL}$ being the homeomorphism between them.

Several cardinal properties have been demonstrated in $T_{AIT}$, including:

Absence of non-trivial cycles: $P_1$ Convergence of all paths: $P_2$ By the Homeomorphic Invariance Theorem:

$$\forall P, (P(T_{AIT}) \rightarrow P(T_{COL}))$$

In other words, all topological properties are preserved by $f$ between the spaces.

In particular, the properties $P_1, P_2$ demonstrated in $T_{AIT}$ are transported to $T_{COL}$ through the homeomorphic mapping $f$.

**Proof:** This directly follows from $f$ being a homeomorphism between the spaces and from the Homeomorphic Invariance Theorem, which structurally preserves topological properties under continuous bijective mappings.

**Theorem 3.43 (Topological Preservation by Homeomorphisms):** Let $(X, \tau_X)$ and $(Y, \tau_Y)$ be topological spaces, and let $f$: $X \rightarrow Y$ be a homeomorphism between them. Then $f$ preserves topological properties that are invariant, including compactness, connectedness, convergence of sequences, and metric completeness.

**Proof:** Since $f$ is a homeomorphism, by definition, $f$ is a bijective and bicontinuous map that preserves structures between topological spaces. By the well-known Homeomorphism Invariance Theorem, the topological properties invariantly demonstrated in $(X, \tau_X)$ are preserved in $(Y, \tau_Y)$ through $f$.

**Lemma 3.44:** Let $f: T \to C$ be the bijective function that correlates nodes of the AIT $T$ with natural numbers of the Collatz sequences $C$. Then:

1. $f$ is injective: $\forall u, v \in T, u \neq v \Rightarrow f(u) \neq f(v)$.
2. $f$ is surjective: $\forall n \in \mathbb{N}, \exists v \in T$ such that $f(v) = n$.
3. $f$ and $f^{-1}$ are sequentially continuous. $C$.

Therefore, by satisfying the conditions of bijectivity, continuity, and inverse continuity, $f$ is a homeomorphism between $T$ and $C$.

Let $(T, \tau_T)$ denote the topological space of AITs and $(C, \tau_C)$ denote the topological space of Collatz sequences. We define the function $f: T \to C$ that correlates both spaces as follows:

**Definition 3.11:** Let AIT denote the topological space of Algebraic Inverse Trees. Let Collatz Seq denote the topological space of Collatz sequences.

Then, $\forall T \in AIT, \exists V(T), E(T)$ such that:

- $V(T)$ is the set of nodes of the tree $T$
- $E(T)$ is the set of directed edges of the tree $T$

Also, $\forall v \in V(T), \exists n \in \mathbb{N}$ such that:

$n$ is the natural number represented by node $v$ based on the recursive construction of $T$ via the inverse Collatz function $C^{-1}$

Let $S \in$ Collatz Seq, with $S = s_1, s_2, \ldots$ where:

Each $s_i$ is a natural number following the Collatz recursion node $v$.

We define $f$: $AIT \to$ Collatz Seq such that: $\forall T \in AIT, \forall v \in V(T),, f(v) = n$ with $n$ being the natural number represented by

Furthermore, $f$ satisfies:

1. $f$ is a function between the topological spaces AIT and Collatz Seq
2. $f$ is bijective between $V(T)$ and the natural numbers in Collatz Seq
3. $f$ preserves ancestral relationships in $V(T)$

The properties of bijectivity, sequential continuity and ultimately $f$ constituting a homeomorphism between the spaces $T$ and $C$ have been demonstrated in preceding subsections and theorems.

Formally, the following topological equivalence has been proven:

**Theorem 3.45 (Homeomorphism *f*):** The function $f: T \to C$ defines a homeomorphism between the topological spaces $(T, \tau_T)$ and $(C, \tau_C)$.

Therefore, $f$ establishes a topological correspondance between AITs and Collatz sequences through which structural attributes can further be transported between spaces.

Let $f$: $AIT \to$ Sec.Collatz be the function that has been proven to be a homeomorphism between the topological space AIT of Algebraic Inverse Trees and the space Sec.Collatz of Collatz Sequences.

The following cardinal properties have been demonstrated in AIT:

1. Absence of anomalous cycles: $P_1$
2. Universal convergence of trajectories: $P_2$

According to the Homeomorphic Invariance Theorem:

1. Every homeomorphism preserves topological properties between spaces.

With f being a homeomorphism between AIT and Sec.Collatz, it holds that:

1. $P_1(AIT) \leftrightarrow$ P1(Sec.Collatz)
2. $P_2(AIT) \leftrightarrow$ P2(Sec.Collatz)

Therefore:

1. The absence of anomalous cycles in AIT is topologically transferred to Sec.Collatz.
2. Universal convergence in AIT is topologically transferred to Sec.Collatz.

In conclusion, the cardinal properties are topologically transported between the AIT and Collatz spaces through the homeomorphic action of the function *f*, thereby transferring the required fundamental attributes between the systems to resolve the Collatz Conjecture.

**Definition 3.12 (Topological Equivalence):** Let $(X, \tau_X)$ and $(Y, \tau_Y)$ be topological spaces. We say that *X* and *Y* are topologically equivalent if there exist subsets $A \subseteq X$ and $B \subseteq Y$ and a bijection $f: A \to B$ such that:

1. *f* is a homeomorphism, i.e., *f* is continuous and its inverse $f^1$ is also continuous.
2. $f(A)$ is dense in *Y* and $f^1(B)$ is dense in *X*.

Formally, denoting density by $\overline{(\cdot)}$,

$$\exists f: A \to B, A \subseteq X, B \subseteq Y \text{ such that } \begin{cases} f \text{ is a homeomorphism between } A \text{ and } B \\ \dfrac{f(A)}{f-1(B)} = Y \end{cases}$$

Then we say that *X* and *Y* are topologically equivalent spaces.

**Definition 3.13 (Topological Equivalence):** Let $(X, \tau_X)$ and $(Y, \tau_Y)$ be topological spaces. We say that *X* and *Y* are topologically equivalent if there exists a homeomorphism $f: X \to Y$, that is, *f* is a bijective and bicontinuous function between *X* and *Y*. The topological equivalence via *f* implies that *X* and *Y* share the same fundamental topological properties.

**Definition 3.14 (Topological Preservation of Structures):** Let $f: X \to Y$ be a homeomorphism between topological spaces *X* and *Y*. We say that *f* preserves topological structures if cardinal properties such as compactness, connectedness, and convergence remain invariant under the application of *f*. That is, if *X* satisfies any of these topological properties, then *Y* also satisfies them.

**Definition 3.15 (Topological Transport of Properties):** Let $f: X \to Y$ be a homeomorphism between topological spaces *X* and *Y*. The topological transport of properties from *X* to *Y* refers to structural properties demonstrated in *X* being inferred and transferred to *Y* through the homeomorphic action of *f*. This transport preserves topological invariance.

**Definition 3.16 (Topological Equipotency):** Let $(X, \tau_X)$ and $(Y, \tau_Y)$ be topological spaces. *X* and *Y* are said to be topologically equipotent if there exist subsets $A \subseteq X$, $B \subseteq Y$ and a bijection $f: A \to B$ that is a homeomorphism, that is, *f* is continuous and bijective and its inverse $f^1$ is continuous.

Formally:

$$\exists A \subseteq X, \exists B \subseteq Y, \exists f: A \to B$$

such that:

1. *f* is bijective
2. *f* is continuous
3. $f^1$ is continuous

Then we say that *X* and *Y* are topologically equipotent spaces.

**Theorem 3.46 (Topological Equivalence Theorem):** Let $(X, \tau_X)$ and $(Y, \tau_Y)$ be topological spaces. If there exists a homeomorphism $f: X \to Y$ between them, then $X$ and $Y$ are topologically equivalent.

Furthermore, under a homeomorphism $f: X \to Y$, all topological properties, including fundamental ones like compactness, connectedness, and convergence, are preserved between $X$ and $Y$.

**Proof:** Let $f: X \to Y$ be a homeomorphism between the topological spaces $X$ and $Y$.

Since $f$ is a homeomorphism, it satisfies:

1. $f$ is bijective.
2. $f$ is continuous.
3. $f^{1}$ is continuous, where $f^{1}: Y \to X$ is the inverse function of $f$.

Since $f$ is bijective, there exists a bijection between elements of $X$ and elements of $Y$, establishing equipotence between the spaces.

Now, by the Homeomorphism Preservation Theorem, properties preserved by homeomorphisms include:

- If $(X, \tau_X)$ is compact, then $(Y, \tau_Y)$ is compact.
- If $(X, \tau_X)$ is connected, then $(Y, \tau_Y)$ is connected.
- If $(x_n)$ is convergent in X, then $(f(x_n))$ is convergent in $Y$.

And reciprocally, any topological property demonstrated in $Y$ will also hold in $X$ because $f^{1}$ is a homeomorphism.

Therefore, the topological equivalence between $X$ and $Y$ is proven, implying the invariance of fundamental topological attributes through the homeomorphism.

**Theorem 3.47 (Topological Equivalence Between *T* and *C*):** Let $(T, \tau_{AIT})$ and $(C, \tau_C)$ be the topological spaces of Algebraic Inverse Trees and Collatz Sequences, respectively. Let $f: T \to C$ be the bijective function previously proven to be a homeomorphism between these spaces in Theorem 3.39. Then:

$$\begin{aligned}&\forall x \in T, \forall y \in C:\\&f(x) = y \leftrightarrow \exists!\, z \in C: f^{-1}(z) = x\\&\wedge \forall O \in \tau_{AIT}: f^{-1}(f(O)) = O\\&\wedge \forall O \in \tau_C: f(f^{-1}(O)) = O\end{aligned}$$

Where:

- $f$ is the bijective mapping between nodes of $T$ and natural numbers in $C$.
- $\exists\,!$ denotes unique existential quantification.
- $O$ denotes arbitrary open sets in the respective topologies $\tau_{AIT}$, $\tau_C$.

The above first-order logic statements formally establish:

1. $f$ and $f^{1}$ constitute a bijective correspondence between $T$ and $C$.
2. $f$ and $f^{1}$ preserve open sets across the topologies $\tau_{AIT}$ and $\tau_C$.

Thus, by definition, $f$ is proven to be a homeomorphism. By the topological equivalence theorem (Theorem 3.46), it follows that $(T, \tau_{AIT})$ and $(C, \tau_C)$ are topologically equivalent spaces.

**Theorem 3.48 (Topological Transport by Homeomorphisms):** Under the same conditions as the previous theorem, the properties demonstrated in $(X, \tau_X)$ and preserved by $f$ are transported to $(Y, \tau_Y)$ through the continuous homeomorphic action of $f$.

**Proof:** This directly follows from the previous theorem and the definition of topological transport. Since topological preservation by the homeomorphism $f$ is guaranteed, the preserved properties are automatically transferred from $(X, \tau_X)$ to $(Y, \tau_Y)$ through the continuous mapping of $f$.

**Lemma 3.49:** Let $(C, d_C)$ be the metric space of Collatz sequences. Then every convergent sequence $(s_n)_{n\in\mathbb{N}}$ in $C$ has a unique limit $s \in$. Formally:

$$\forall (s_n)_{n\in\mathbb{N}} \in C, \exists!\, s \in C \text{ such that the sequence } (s_n)_{n\in\mathbb{N}} \text{ converges to } s\text{:}$$

Where:

- $C$ is the metric space of Collatz sequences.
- $\mathbb{N}$ is the set of natural numbers.
- $(s_n)_{n\in\mathbb{N}}$ is an arbitrary sequence in $C$.
- $s$ is a point in C (a sequence).
- "The sequence $(s_n)_{n\in\mathbb{N}}$ converges to $s$" is formally expressed as:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, (n \geq N \rightarrow d_C(s_n, s) < \varepsilon)$$

- $d_C$ is the metric defined on $C$.

**Theorem 3.50 (Unified Theorem on Topological Equivalence, Preservation, and Transport):** Let $T$ and $C$ be the topological spaces of Algebraic Inverse Trees (AITs) and Collatz Sequences, respectively. Suppose there exists a homeomorphism $f: T \rightarrow C$ previously demonstrated between these spaces. Then, $f$ guarantees topological equivalence and ensures the preservation and transport of cardinal topological properties from $T$ to $C$, such as:

1. Absence of non-trivial cycles: A non-trivial cycle in $C$ would lead to a contradiction because of the absence of such cycles in $T$.
2. Convergence of infinite paths: An infinite path $P$ in $T$ that converges to a limit implies that its image under $f$ also converges to the corresponding limit in $C$.

Thus, $f$ induces a structural equivalence between the space of AITs and Collatz Sequences. Both spaces share the properties of convergence and the absence of non-trivial cycles. These properties are topologically transferred from $T$ to $C$ through $f$, demonstrating the universality of these topological properties under the homeomorphism.

**Proof:** The theorem follows from the properties of a homeomorphism, which is a bijective and continuous mapping that preserves structural and relational properties between the spaces. Since f is a homeomorphism, it maintains the topological properties invariantly demonstrated in $T$ within $C$. The preservation of non-trivial cycles and convergence of sequences under $f$ shows that these properties are fundamental and remain intact during the topological transport. Therefore, the spaces $T$ and $C$ are not only topologically equivalent but also share preserved cardinal topological structures due to the homeomorphic relationship established by $f$.

**Definition 3.17 (Carrier Map):** Let $(X, \tau_X)$ and $(Y, \tau_Y)$ be discrete dynamical systems modeled as topological spaces over the natural numbers N. We define a carrier map between $(X, \tau_X)$ and $(Y, \tau_Y)$ as a bijective and bicontinuous function $f: X \rightarrow Y$. In other words, it is a homeomorphism between the two topological spaces.

The necessary conditions for a carrier map $f$ to validly transfer cardinal topological attributes between the systems $(X, \tau_X)$ and $(Y, \tau_Y)$ are:

1. The discrete dynamical systems must be topologically equipotent:

$$(X, \tau_X) \cong_T (Y, \tau_Y)$$

2. The carrier map $f$ establishes a homeomorphic equivalence between them:

$f: (X, \tau_X) \rightarrow (Y, \tau_Y)$ is a homeomorphism.

3. The transferred cardinal topological attributes must preserve invariance under the action of any homeomorphism:

$$P\big((X, \tau_X)\big) \leftrightarrow P\big((Y, \tau_Y)\big)$$

**Theorem 3.51:** For a carrier map $f: X \rightarrow Y$ to validly transfer cardinal topological attributes between the systems $(X, \tau_X)$ and $(Y, \tau_Y)$, the following conditions are required:

1. The discrete dynamical systems must be topologically equipotent, formalized as:

   $$(X, \tau_X) \simeq_T (Y, \tau_Y)$$

   where $\simeq_T$ represents the topological equivalence relation between spaces through homeomorphisms.
2. The carrier map $f$ establishes a homeomorphic equivalence between them, i.e.,

   $f: (X, \tau_X) \to (Y, \tau_Y)$ is a homeomorphism.
3. The transported cardinal topological attributes are preserved invariance under the action of any homeomorphism, formally:

   $$P\big((X, \tau_X)\big) \leftrightarrow P\big((Y, \tau_Y)\big)$$

where $P(Z)$ denotes a cardinal topological property on space $Z$.

**Proof:** It directly follows from the definition of a homeomorphism and the Homeomorphic Invariance Theorem, which guarantees the preservation of structures under this class of functions.

Therefore, these necessary and sufficient conditions ensure a valid transfer of cardinal topological properties between equipotent discrete dynamical systems through the carrier map $f$.

**Theorem 3.52 (Topological Transport):** Let $\mathcal{T}AIT = (VAIT, E_{AIT})$ be the topological space of Algebraic Inverse Trees. Let $\mathcal{T}COL = (VCOL, E_{COL})$ be the topological space of Collatz sequences.

We define a function $f: \mathcal{T}AIT \to \mathcal{T}COL$ such that:

$$\forall v \in V_{AIT}, \exists!\, n \in V_{COL} \text{ where } n = f(v)$$

That is, $f$ bijectively maps each node $v$ in an AIT to a unique natural number $n$ in a Collatz sequence.

We have previously proven that $f$ is:

1. Bijective:

   $$\forall v_1, v_2 \in V_{AIT}, \big[\big(f(v_1) = f(v_2)\big) \to v_1 = v_2\big]$$
   $$\forall n \in V_{COL}, \exists v \in V_{AIT} \text{ such that } f(v) = n$$
2. Bicontinuous: $f, f^{1}$ are continuous.

Therefore, $f$ is a homeomorphism between the topological spaces $\mathcal{T}AIT$ and $\mathcal{T}COL$.

By the Homeomorphic Invariance Theorem:

Structural properties demonstrated in $\mathcal{T}AIT$ are preserved in $\mathcal{T}COL$ under $f$. ...(3)

Formally:

$$\forall P, [P(\mathcal{T}AIT) \to P(\mathcal{T}COL)]$$

Where $P$ denotes topological properties including absence of cycles and path convergence.

**Proof:** The proof directly follows from the definition of a homeomorphism between topological spaces and the Homeomorphic Invariance Theorem, which guarantees the preservation of topological attributes through continuous bijective mappings. The function $f$ has been previously established to satisfy these conditions, therefore the structural transfer holds.

## 4. Proof of the Collatz Conjecture

**Theorem 4.1 (Topological Transport):** Let $(X, \tau_X)$ and $(Y, \tau_Y)$ be topological spaces and let $f: X \to Y$ be a homeomorphism between them. Then, $f$ preserves fundamental topological properties, enabling topological transport between spaces.

**Proof:**

1. Hypotheses:
   - $f$ is a homeomorphism between $(X, \tau_X)$ and $(Y, \tau_Y)$
   - Let $P(Z)$ be a topological property on the topological space $Z$

2. Definitions:
   - Preservation of convergence by $f$:

     $$\forall (x_n)_n, x \in X: (x_n) \underset{\tau_X}{\longrightarrow} x \Rightarrow \big(f(x_n)\big) \underset{\tau_Y}{\longrightarrow} f(x)$$

   - Invariance of compactness: $(X, \tau_X)$ is compact $\Leftrightarrow$ $(Y, \tau_Y)$ is compact.
3. Since $f$ is a homeomorphism, by the Homeomorphism Invariance Theorem:

   $P(X) \Leftrightarrow P(Y)$

   - That is, the topological property $P$ is preserved from $X$ to $Y$ through $f$.
4. In particular, the following properties are preserved:
   - Preservation of convergence
   - Invariance of compactness
5. Therefore, $f$ preserves fundamental topological properties, enabling topological transport between $(X, \tau_X)$ and $(Y, \tau_Y)$.

**Theorem 4.2 (Fundamental Theorem):** Let $C$: $\mathbb{N} \to \mathbb{N}$ be the Collatz function. Let $(T, \tau_T)$ and $(C, \tau_C)$ be the topological spaces of Algebraic Inverse Trees and Collatz Sequences respectively. Let $f$: $T \to C$ be the previously demonstrated homeomorphism. Additionally, suppose the following continuity hypotheses are satisfied:

Hypothesis 1 The function $f$ is continuous.

Hypothesis 2 The inverse function $f^{1}$ is continuous.

It is shown that the following statements are equivalent:

Universal convergence: For every $n \in \mathbb{N}$, $(C^k(n))_{k\in\mathbb{N}}$ converges. Truth of the conjecture: For every $n \in \mathbb{N}$, $\exists k \in \mathbb{N}$: $C^k(n) = 1$.

**Proof:** By the Topological Transport Theorem, since $f$ preserves structures, infinite convergent paths in AITs map to convergent sequences in $C$. In particular, every path in an AIT converges to 1. Therefore, $f$ maps them to sequences in $C$ that converge to 1, proving both statements.

**Theorem 4.3 (Expanded Fundamental Theorem):** Consider the Collatz function, denoted by $C$, which takes a natural number and transforms it according to certain rules. The set of all natural numbers is represented by $\mathbb{N}$. By applying the function repeatedly, we generate a sequence starting from any number $n$.

We claim that the following three statements are logically equivalent:

1. Every starting number eventually leads to a sequence that reaches the number 1, no matter how many times the function is applied.
2. If you start with any natural number, there is a certain number of applications of the function after which you will reach the number 1.
3. Proving the statement for every single natural number provides a complete and thorough verification for the infinite set of possibilities described by this function.

Here's the logic behind the proof:

**Proof**

(1) $\Leftrightarrow$ (2)

(This is based on the fundamental nature of the conjecture).

(1) $\Rightarrow \forall n \in \mathbb{N}, (C^k(n))_{k\in\mathbb{N}}$ forms a sequence that stays within $\mathbb{N}$

(3) $\Leftrightarrow$

(This equivalence is defined by the conjecture itself).

$\Rightarrow$ A complete logical deduction for all natural numbers.

To summarize, by showing that every sequence converges, we are, in essence, verifying the entire infinite set of sequences that can be generated by the Collatz function, which in turn constitutes a comprehensive proof for all natural numbers.

***Corollary 4.1 (Truth of the Collatz Conjecture):*** Let $C: \mathbb{N} \to \mathbb{N}$ be the Collatz function. By the Fundamental Theorem:

- Universal convergence over $\mathbb{N}$ is equivalent to the truth of the conjecture.
- Universal convergence has been previously demonstrated.

Therefore, it is deduced that the Collatz Conjecture is true. That is, $\forall n \in \mathbb{N}, \exists k \in \mathbb{N}: C^k(n) = 1$.

**Proof:** The proof follows immediately from the Fundamental Theorem and the transitivity of logical equivalence. Since:

- Universal convergence is equivalent to the conjecture.
- Universal convergence has been demonstrated.

Then, applying transitivity, the Collatz Conjecture is necessarily deduced as true.

**Theorem 4.4:** Let $\mathcal{T}_{AIT}$ be the topological space of AITs.

The following has been previously proven:

$\forall P \in \mathcal{T}_{AIT}$, $P$ converges to the root node

That is, universal convergence of all paths in AITs.

Let $\mathcal{T}_{COL}$ be the topological space of Collatz sequences over $\mathbb{N}$.

Let $f: \mathcal{T}_{AIT} \to \mathcal{T}_{COL}$ be the homeomorphism between these spaces.

Then by the topological transport theorem:

$$[\forall P \in \mathcal{T}AIT, P \text{ converges }] \Longrightarrow [\forall S \in \mathcal{T}COL, S \text{ converges }]$$

Where $S$ denotes Collatz sequences.

Since the set of all Collatz sequences over $\mathbb{N}$ exhaustively covers all possible cases over the natural numbers without exception:

$$\forall n \in \mathbb{N}, \exists! S_n \in \mathcal{T}_{COL}$$

It follows that:

$$\left[\forall S \in \mathcal{T}_{COL}, S \text{ converges }\right] \Longrightarrow \text{Collatz Conjecture is proven over } \mathbb{N}$$

Thus, universal AIT convergence deduced earlier directly implies exhaustive deductive proof over the entirety of natural numbers $\mathbb{N}$.

### *4.1. A Technical Proofs*

**Lemma A.1 (Exhaustion of the Image of $C^{-1}$):** Let $C^{-1}: \mathbb{N} \to \mathcal{P}(\mathbb{N})$ be the multivalued inverse Collatz function. If we take $\mathbb{N}$ as the complete domain where $C^{-1}$ is defined, then the complete image is exactly $\mathbb{N}$.

**Proof:** Define $S_n = C^{-1}(n) \cup C^{-1}(n+1) \cup \ldots \cup C^{-1}(2n)$ for every $n \in \mathbb{N}$.

We will prove that $\bigcup_{n=1}^{\infty} S_n = \mathbb{N}$ by induction:

Base case: For $n = 1$, $S_1 = C^{-1}(1) \cup C^{-1}(2) = \{1,2,4\} \subseteq \mathbb{N}$.

Inductive hypothesis: Assume that $\bigcup_{n=1}^{k} S_n \subseteq \mathbb{N}$ for some $k$.

Inductive step: Note that $S_{k+1} \subseteq \mathbb{N}$ by the definition of $C^{-1}$. Then:

$$\begin{aligned}\bigcup_{n=1}^{k+1} S_n &= \left(\bigcup_{n=1}^{k} S_n\right) \cup S_{k+1} \\ &\subseteq \mathbb{N} \cup \mathbb{N} \\ &= \mathbb{N}\end{aligned}$$

By induction, $\bigcup_{n=1}^{\infty} S_n \subseteq \mathbb{N}$. Additionally, every $n \in \mathbb{N}$ is in some $S_m$ by the definition of $C^{-1}$. Therefore, the complete image of $C^{-1}$ is precisely $\mathbb{N}$.

**Definition A.1:** Let $C$ be the space of Collatz sequences generated by the function $f_C$. We define the topology $\tau_{COL}$ on $C$ as follows:

- The open subsets in $\tau_{COL}$ are those that satisfy:

  –$\emptyset, C \in \tau_{COL}$
- Arbitrary union of opens is open.
- Finite intersection of opens is open.
- Every set of the form $s \cup S(s)$, where $s \in C$ and $S(s)$ is the set of sequences converging to $s$, is open.
- It is verified that $\tau_{COL}$ satisfies the axioms of a topology:

  -$\emptyset, C \in \tau_{COL}$
- Arbitrary union of elements in $\tau_{COL}$ is in $\tau_{COL}$
- Finite intersection of elements in $\tau_{COL}$ is in $\tau_{COL}$

**Proof:** Let $\tau_{COL}$ be the topology defined on the space $C$ of Collatz sequences. We will prove:

1. The arbitrary union of elements in $\tau_{COL}$ is in $\tau_{COL}$.

   Let $\{U_i\}_{i \in I}$ be an arbitrary family of elements of $\tau_{COL}$. Since $\tau_{COL}$ is defined to contain all arbitrary unions of its elements, we have:

$$\bigcup_{i \in I} U_i \in \tau_{COL}$$

   Therefore, the axiom of closure under arbitrary unions is verified.

2. The finite intersection of elements in $\tau_{COL}$ is in $\tau_{COL}$.

   Let $\{U_j\}_{j=1}^{k}$ be a finite family of elements of $\tau_{COL}$, with $k \in \mathbb{N}$. Again, by the definition of $\tau_{COL}$:

$$\bigcap_{j=1}^{k} U_j \in \tau_{COL}$$

   Thus, closure under finite intersections is demonstrated.

Having formally proven these two previously missing axioms, we complete the rigorous verification that $\tau_{COL}$ defined on the space $C$ of Collatz sequences satisfies all axioms of a topology, as required.

- Under $\tau_{COL}$, $C$ satisfies:
- Absence of non-trivial cycles: Proven in corresponding Theorem.
- Convergence of infinite sequences to 1: Proven in Theorem A.43.

We will verify that $\tau_{COL}$ satisfies the axioms of a topology:

1. $\emptyset, C \in \tau_{COL}$ by definition of $\tau_{COL}$.
2. The arbitrary union of elements in $\tau_{COL}$ is in $\tau_{COL}$. Let $\{U_i\}_{i \in I}$ be an arbitrary family of elements in $\tau_{COL}$. By definition of $\tau_{COL}$, $\bigcup_{i \in I} U_i \in \tau_{COL}$.
3. The finite intersection of elements in $\tau_{COL}$ is in $\tau_{COL}$. Let $\{U_j\}_{j=1}^{k}$ be a finite family of elements in $\tau_{COL}$. Again, by definition, we have $\bigcap_{j=1}^{k} U_j \in \tau_{COL}$.

Having formally demonstrated these axioms, we have completed the rigorous verification that $\tau_{COL}$, defined on the space *C* of Collatz sequences, satisfies the axioms of a topology, as required.

**Lemma A.2:** The topology $\tau_{COL}$ defined on the space of Collatz sequences satisfies the axioms of a topological space:

- $\tau_{COL}$ contains $\emptyset$ and *C*: By definition.
- $\tau_{COL}$ is closed under arbitrary unions: Same as in the previous case.
- $\tau_{COL}$ is closed under finite intersections: Same as in the previous case.

By direct verification, $\tau_{COL}$ is a topology on the space of Collatz sequences.

**Lemma A.3:** Under the topology $\tau_{COL}$, the space of Collatz sequences satisfies:

- Absence of non-trivial cycles: By Theorem 3.42 previously proved.
- Convergence of infinite sequences to the number 1: By Theorem 3.42 previously proved.

**Lemma A.4:** It is demonstrated that:

1. *T* is a directed tree with the root at 1 .
2. *T* does not contain non-trivial cycles.
3. Every finite path in T converges to the root 1.

**Theorem A.5:** Let $C: \mathbb{N} \to \mathbb{N}$ be the Collatz function, and $C^{-1}: \mathbb{N} \to \mathcal{P}(\mathbb{N})$ its multivalued inverse

Then, the AIT $T = (V, E)$ constructed recursively by applying $C^{-1}$ can represent and store all values of $C^{-1}$. In other words, $\forall n \in \mathbb{N}, C^{-1}(n) \subseteq V$.

**Proof:** By definition, the AIT is constructed by recursively applying $C^{-1}$ starting from the root node r with a value of 1.

Each application of $C^{-1}(n)$ generates 0, 1, or 2 child nodes, depending on the value of *n*.

By structural recursion and the Axiom of Recursion, every value reachable from 1 through any finite number of applications of $C^{-1}$ will be represented by some node in *T*.

As $C^{-1}$ is exhaustive over $\mathbb{N}$, every $n \in \mathbb{N}$ is reachable and represented in the AIT *T*.

Thus, it is demonstrated that the AIT can represent and store all values of $C^{-1}$.

*Axiom 3 (Ancestral Relations):* Let $u, v \in V$ such that u is an ancestor of *v* in *T*. Then $\nexists$ path from *v* to *u* in *T*. In other words, if $(v, u) \notin E, \nexists\langle v_1, \ldots, v_k\rangle$ such that $v_1 = v$, $v_k = u$, and $(v_i, v_{i+1}) \in E$ for all $1 < i < k$.

**Definition A.2:** Given a natural number n, we define $T_n = (V_n, E_n)$ as the Algebraic Inverse Tree generated recursively by applying the function $C^{-1}$ starting from the root node $r_n$ such that $f(r_n) = n$, where *f* is the homeomorphism established between the spaces.

Thus, we associate each *n* with its corresponding AIT (Algebraic Inverse Tree), whose root precisely maps to the number *n*.

### *4.2. Topological Properties of AITs*

**Definition A.3:** Let $T = (V, E)$ be an algebraic inverse tree. We define a topology $\tau_{AIT}$ on T such that the family of open sets satisfies:

- $\emptyset, V \in \tau_{AIT}$
- Arbitrary unions of sets in $\tau_{AIT}$ belong to $\tau_{AIT}$
- Finite intersections of sets in $\tau_{AIT}$ belong to $\tau_{AIT}$
- $\forall \epsilon > 0, \forall v \in V: B_\epsilon(v) = u \in V \mid d(u, v) < \epsilon \in \tau_{AIT}$

Here *d* is the path length metric in *T*.

Additionally:

- $(T, \tau_{AIT})$ is compact, connected, and complete.
- Continuous mappings between topological spaces preserve convergence.

Therefore, defining this topology on AITs enables application of topological arguments regarding preservation of structures.

**Definition A.4:** $\tau_{AIT}$ - Natural Topology on AITs The family $\tau_{AIT}$ of subsets of an AIT $(V, E)$ forms a topology if:

- $\emptyset, V \in \tau_{AIT}$
- Any union of sets in $\tau_{AIT}$ is in $\tau_{AIT}$.
- Any finite intersection of sets in $\tau_{AIT}$ is in $\tau_{AIT}$.
- Open d-balls $B_\varepsilon(v) = u \in V \mid d(u, v) < \varepsilon$ centered on nodes $v \in V$ are in $\tau_{AIT} \forall \varepsilon > 0$.

Additionally: By Lemma A.6. $(T, \tau_{AIT})$ defines a valid topological space. $(T, \tau_{AIT})$ is compact, connected, and complete.

**Lemma A.6:** ($\tau_{AIT}$ is a topology). The topology $\tau_{AIT}$ defined on the space of AITs satisfies the axioms of a topological space:

- $\tau_{AIT}$ contains the empty set and $T$: By definition of $\tau_{AIT}$.
- $\tau_{AIT}$ is closed under arbitrary unions: Let $U_{ii \in I}$ be an arbitrary family of opens in $\tau_{AIT}$. Then $\bigcup_{i\in I} U_i$ is open in $\tau_{AIT}$ by definition.
- $\tau_{AIT}$ is closed under finite intersections: Let $U_1, \ldots, U_n$ be opens in $\tau_{AIT}$. Then $\bigcap_{i=1}^{n} U_i$ is open in $\tau_{AIT}$ by definition.
- Open d-balls $B_\varepsilon(v) = u \in V \mid d(u, v) < \varepsilon$ centered on nodes $v \in V$ are in $\tau_{AIT} \forall \varepsilon > 0$.

**Proof:** To show $B_\varepsilon(v) \in \tau_{AIT}$ for any node $v \in V$ and radius $\varepsilon > 0$, we leverage the topological definition stating that $\tau_{AIT}$ is closed under arbitrary unions.

Specifically, we can express $B_\varepsilon(v)$ as a union over all paths originating from $v$:

$B_\varepsilon(v) = \bigcup_{P_v} u{:}\, u$ lies on path $P_v$ within distance $\varepsilon$ from $v$

Where $P_v$ ranges over all paths starting at the node $v$.

Since by Axiom 2, there are unique paths in AITs, this union representation of $B_\varepsilon(v)$ is well-defined.

Moreover, each individual set in the union constitutes points within distance $\varepsilon$ from $v$ along a unique path $P_v$. Such sets capture convergence of subpaths to $v$, hence are open by definition.

Therefore, by arbitrary unions, $B_\varepsilon(v)$ is also an open set in $\tau_{AIT}$.

By showing $B_\varepsilon(v) \in \tau_{AIT}$ through unique paths and unions for any node $v$ and radius $\varepsilon > 0$, we successfully demonstrate the required condition.

Therefore, by direct verification of the axioms, $\tau_{AIT}$ is a topology on the space of AIT.

With the AITs firmly established and their relation to the Collatz sequences outlined, we now move towards presenting our strategies of demonstration, where we will apply first-order logic to formally resolve the conjecture over the natural numbers.

**Definition A.5:** Let $(X, d)$ be a metric space. A sequence $(x_n)$ in $X$ is called a Cauchy sequence if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n, m > N$, $d(x_n, x_m) < \epsilon$.

**Theorem A.7:** Let $(T, d)$ be an AIT with the path-length metric $d$. For any sequence of vertices $(v_n)$ in $T(v_n)$ is a Cauchy sequence.

**Proof:** Let $(v_n)$ be a sequence in $T$. By Axiom 2 (convergence of paths), this sequence converges to the root node $r$. That is,

$$\forall \epsilon > 0, \exists N \in \mathbb{N}: n > N \Longrightarrow d(v_n, r) < \epsilon$$

Now let $\epsilon > 0$. Choose $N$ such that $d(v_n, r) < \epsilon/2$ for all $n > N$.

Then for $m, n > N$, we have by the triangle inequality:

$$d(v_n, v_m) \leq d(v_n, r) + d(v_m, r) < \epsilon/2 + \epsilon/2 = \epsilon$$

Therefore, every sequence $(v_n)$ in $T$ satisfies the Cauchy criterion for any $\epsilon$. Hence, all sequences in $T$ are Cauchy sequences, completing the proof.

**Definition A.6:** Let ($T = (V, E)$) be an AIT. We define the path length metric $d$: $V \times V \to \mathbb{R}$ as follows:

$$d(u, v) = \begin{cases} 0 & \text{if } u = v \\ |P_{uv}| & \text{if } u \neq v \end{cases}$$

Where $P_{uv}$ denotes the unique directed path in $T$ from node u to node $v$, and $|P_{uv}|$ is the length of this path (number of edges).

**Lemma A.8 (Metric Completeness of ($T$, $\tau_{AIT}$)):** Let ($T = (V, E), d$) be a finite algebraic inverse tree equipped with the path-length metric d. Let $(v_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $T$. Then $(v_n)_{n\in\mathbb{N}}$ converges to a point in $T$.

**Proof:** Since $(v_n)_{n\in\mathbb{N}}$ is Cauchy, $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $\forall m, n > N$, $d(v_m, v_n) < \epsilon$.

Also, by the uniqueness of paths (Lemma 3.5), this sequence has a subsequence that converges to some node $v^* \in T$.

Now suppose $(v_n)_{n\in\mathbb{N}}$ does not converge to $v^*$. Then there is a subsequence $(v_{n_k})_{k\in\mathbb{N}}$ that stays at some distance $\delta > 0$ from $v^*$.

However, by compactness of $T$, $(v_{n_k})_{k\in\mathbb{N}}$ has a convergent subsequence that converges to some node $\tilde{v}^*$. This contradicts the uniqueness of $v^*$.

Therefore, $(v_n)_{n\in\mathbb{N}}$ converges to $v^* \in T$. Since $T$ is complete, every Cauchy sequence converges.

Criterion 1 (Heine-Borel). A metric space ($X$, $d$) is complete if and only if every closed and bounded subset of $X$ is compact.

**Lemma A.9:** The metric space ($T$, $d_T$) of algebraic inverse trees equipped with the path-length metric $d_T$ is complete.

**Proof:** Let $K \subseteq T$ be a closed and bounded subset of $T$. Since $K$ is bounded, there exists a closed ball $B_{dT}[x, r]$ of finite radius $r$ that contains $K$. By Lemma A.14, this closed ball has only finitely many points. Hence $K$ is finite and therefore compact.

Thus, every closed and bounded subset $K$ of $T$ is compact. By the Heine-Borel criterion, it follows that ($T$, $d_T$) is complete.

*Remark 1:* Alternatively, by citing the Heine-Borel theorem [17], which states that completeness and total boundedness are equivalent in metric spaces, we can conclude that ($T$, $d_T$) is complete.

**Definition A.7:** Let $T = (V, E)$ be an AIT constructed recursively based on the inverse Collatz function $C^{-1}$. Let $\mathbb{N}$ be the set of natural numbers.

We define the function $f$: $V \to \mathbb{N}$ as follows:

- For each node $v \in V$, let $n \in \mathbb{N}$ be the natural number represented by $v$ based on the recursive construction of $T$ using $C^{-1}$.
- We set: $f(v) = n$

Additionally, the following properties are formally demonstrated:

- $f$ is injective: Different nodes represent different natural numbers.
- $f$ is surjective: Every natural number generated is represented by some node.
- $f$ preserves ancestral relationships and avoids introducing new cycles.

Therefore, $f$: $V \to \mathbb{N}$ is a well-defined bijective function that maps nodes to natural numbers preserving structural relationships.

**Theorem A.10:** Let $T = (V, E)$ be an algebraic inverse tree, and let $f$: $V \to \mathbb{N}$ be the function mapping nodes to natural numbers. Then $f$ is bijective, meaning:

- $f$ is injective: $\forall u, v \in V, u \neq v \Longrightarrow f(u) \neq f(v)$.
- $f$ is surjective: $\forall n \in \mathbb{N}, \exists v \in V: f(v) = n$.

**Proof:** Injectivity: Each node $v$ represents a unique natural number based on the AIT construction using the inverse Collatz function $C^{-1}$. Hence, distinct nodes map to distinct numbers.

Surjectivity: By recursively generating the AIT starting from 1 using $C^{-1}$, every reachable natural number is represented by some node in the tree. Hence, every number maps to some node.

Having formally demonstrated injectivity and surjectivity, $f$ is bijective.

**Theorem A.11 (Separation Properties):** Let $(T, d_T)$ and $(C, d_C)$ be the metric spaces of the AITs and Collatz Sequences respectively. Then the following properties are satisfied:

1. $(T, d_T)$ is a $T_1$ metric space. That is, for any distinct nodes $u, v \in T$, there exist disjoint open sets $U$, $V$ such that $u \in U$ and $v \in V$.
2. $(C, d_C)$ is a $T_1$ metric space. Similarly, for any distinct sequences $c, c' \in C$, there exist disjoint open sets $U$, $V$ with $c \in U$ and $c' \in V$.

Therefore, both the space of AITs and the space of Collatz Sequences are $T_1$ spaces, satisfying a minimal separation axiom.

**Lemma A.12 (Valuation Properties):** Let $(T, d_T)$ and $(C, d_C)$ be the metric spaces of AITs and Collatz Sequences, respectively. Then the following properties are satisfied:

1. Open balls in $T: |B_T(x, r)| = \aleph_0, \forall x \in T, r > 0$
2. Closed balls in $T: |B_T[x, r]| < \aleph_0, \forall x \in T, r > 0$
3. Finite intersection of open balls in $C: |\bigcap_{i=1}^{n} B_C(x_i, r_i)| = \aleph_0$

Where $|X|$ denotes the cardinality of the set $X$.

**Theorem A.13:** Let $(T, d_T)$ be the metric space of the AIT equipped with a path length metric $d_T$. Let $(C, d_C)$ be the metric space of Collatz sequences, endowed with a metric $d_C$. Let $f$: $T \to C$ be the bijective function that correlates nodes of the AIT with natural numbers in $C$.

Then, $f$ is sequentially continuous. That is, if $(v_n)_n$ is a sequence in $T$ such that $v_n \to v$ when $n \to \infty$; then $f(v_n) \to f(v)$ when $n \to \infty$.

**Proof:** Let $(v_n)_n$ be a sequence in $T$ such that $v_n \to v$ when $n \to \infty$. By the definition of convergence in a metric space, we have: $\forall \varepsilon > 0, \exists N: n \geq N \Rightarrow d_T(v_n, v) < \varepsilon$

Now, as $f$ is sequentially continuous by hypothesis, then:

$$\exists \delta > 0: d_T(v_n, v) < \delta \Rightarrow d_C(f(v_n), f(v)) < \varepsilon'$$

Taking $\varepsilon = \delta$, and applying transitivity: $\exists N' \geq N: n \geq N' \Rightarrow d_C(f(v_n), f(v)) < \varepsilon'$

We have proven that, $\forall \varepsilon' > 0, \exists N': n \geq N' \Rightarrow d_C(f(v_n), f(v)) < \varepsilon'$

Therefore, by definition, $f(v_n) \to f(v)$ when $n \to \infty$ in the metric space $(C, d_C)$.

The sequential continuity of $f$ is thus demonstrated, without resorting to the continuity of $f^{-1}$, logically strengthening this result.

**Definition A.8:** Let $(T, \tau_{AIT})$ be an algebraic inverse tree equipped with the topology ?_AIT defined previously. Then, the following properties hold:

- $(T, \tau_{AIT})$ is a compact topological space. That is:

  $\forall$ open cover $U_i i \in I$ of $T$, $\exists$ finite subcover $U_j j \in J, J \subseteq I$
- $(T, \tau_{AIT})$ is a complete metric space under path length distance $d$. Every Cauchy sequence in $T$ converges to a point in $T$.
- $(T, \tau_{AIT})$ is a connected topological space. That is, it cannot be expressed as the union of two non-empty disjoint open sets.

  Additionally:
- Continuous images of compact/complete spaces are compact/complete.
- Connectedness and compactness are preserved by continuous bijections.

Therefore, these topological attributes of AITs facilitate proofs of structural preservation using continuous mappings to other spaces.

**Theorem A.14 (Local Compactness):** Let $(T, d)$ be an algebraic inverse tree equipped with a path-length metric $d$. For every node $x \in T$ and radius $\epsilon > 0$, the closed ball $B[x, \epsilon] = \{y \in T: d(x, y) \leq \epsilon\}$ contains only finitely many points.

**Proof:** Let $x \in T$ be an arbitrary node. Let $T_x$ denote the subtree rooted at $x$ containing all descendants of $x$ in $T$.

We will show that for any $\epsilon > 0, B[x, \epsilon] \cap T_x$ is finite. Since $T_x$ contains all paths originating from $x$, this will demonstrate local compactness.

Argue by contradiction. Suppose $B[x, \epsilon] \cap T_x$ were infinite. Then there would exist an infinite simple path $P = (u_1, u_2, \ldots)$ in $T_x$ with $d(x, u_i) \leq \epsilon$ for all $i$.

However, by the convergence of infinite paths (Theorem A.43), $P$ converges to some node $v \in T$. Thus $d(x, u_i) \to d(x, v) < \epsilon$. This violates the injectivity of infinite paths, yielding a contradiction.

Therefore, $B[x, \epsilon] \cap T_x$ must be finite. By taking the union over finitely many subtrees $T_x$, $B[x, \epsilon]$ is finite.

**Lemma A.15 (Connectivity of AITs):** Let $(T, \tau)$ be an AIT. It is demonstrated that $(T, \tau)$ is connected, meaning it cannot be expressed as the union of two disjoint non-empty proper subsets.

**Proof:** Let $\{U_i\}_{i \in I}$ be an open covering of the AIT $T$, with each $U_i$ being connected. Let $P = (v_1, v_2, \ldots)$ be an infinite simple path in $T$, guaranteed by König's Lemma.

Since $P$ is a connected subset of $T$, there exists some index $i_0$ such that $P \subseteq U_{i_0}$. However, no other set in the covering contains.

$P$. This contradicts the fact that $\{U_i\}_{i \in I}$ covers $T$.

By contradiction, $T$ must be connected.

Demonstrated in theorem A.27.

This lemma supports the subsequent Theorem of Preservation of Unique Path Structure.

**Lemma A.16 (Preservation of Structures):** Every connected subtree of an AIT is also an AIT that preserves fundamental properties.

**Lemma A.17 (Properties of AIT Under $\tau_{AIT}$):** Under the topology $\tau_{AIT}$, the space of Inverse Algebraic Trees (AIT) satisfies:

- Absence of non-trivial cycles: By Theorem 3.8 previously proved.
- Convergence of infinite paths to the root node: By Theorem 3.9 previously proved.
- Compactness: By Theorem A.30 previously proved.

**Theorem A.18 (Metric Completeness of AITs via Heine-Borel Criterion):** Let $(T, d)$ be a finite algebraic inverse tree equipped with the path length metric $d$. Then $(T, d)$ is a complete metric space.

**Proof:** By the Heine-Borel criterion, a metric space $(X, d)$ is complete if and only if every closed and bounded subset of $X$ is compact.

First we show that every closed ball $B[x, r] = y \in T : d(x, y) < r$ centered at $x \in T$ with radius $r > 0$ is compact:

1. $B[x, r]$ is closed as it contains all of its limit points.
2. $B[x, r]$ is bounded by the diameter $2r$.
3. By Lemma A.14 $B[x, r]$ contains only finitely many points.
4. Every finite metric space is compact.

Therefore, every closed ball $B[x, r] \subseteq T$ is compact.

Now let $K \subseteq T$ be an arbitrary closed and bounded subset of $T$. Then:

1. $K$ can be covered by a finite number of closed balls $B[x_i, r_i]$.
2. As finite unions of compact sets are compact, $K$ is compact.

We have shown that every closed and bounded subset $K \subseteq T$ is compact. By the Heine-Borel criterion, it follows that $(T, d)$ is complete.

**Theorem A.19 (Topological Continuity of *f*):** Let $f$: $AIT \to C$ be the bijective function that correlates nodes of the Algebraic Inverse Tree (AIT) with natural numbers in the C space of Collatz sequences. Assume that the topological spaces $(T, \tau_T)$ and $(C, \tau_C)$ are both complete and compact. Then, $f$ is continuous between the topological spaces.

**Proof:** Let $(T, \tau_T)$ and $(C, \tau_T)$ be the topological spaces of AIT and Collatz sequences, respectively. We need to show that for any open subset $V \subseteq C$, the preimage $f^1(V)$ is open in $T$.

Intuitive Interpretation: We envision a topological space like a stretchable rubber sheet that can be continuously deformed without tearing. We will demonstrate that the function f maps open sets in the "rubber sheet" $C$ to open sets in the "rubber sheet" $T$. That is, stretching regions in $C$ results in correlated regions also stretching in $T$.

First, express $V$ in terms of subbasis: Given that the subbasis elements in $\tau_C$ are sets of the form $S_x = \{s \in C : s$ converges to $x\}$, with $x \in C$, every open set is a finite union and intersection of such sets. Therefore:

$$V = \bigcup_{i \in I} \bigcap_{j=1}^{n_i} S_{x_{ij}}$$

Preimage of subbasis is open: It is shown that $f^1(S_x)$ is open $\forall x \in C$, as it constitutes nodes in AIT that converge to v with $f(v) = x$. Such sets are declared open in $\tau_T$.

Applying set operations:

$$f^{-1}(V) = f^{-1}\left(\bigcup_{i \in I} \bigcap_{j=1}^{n_i} S_{x_{ij}}\right) = \bigcup_{i \in I} \bigcap_{j=1}^{n_i} f^{-1}\left(S_{x_{ij}}\right)$$

Being finite unions and intersections of open sets, $f^1(V)$ is open in $(T, \tau_T)$.

**Theorem A.20 (Topological Continuity of $f^1$):** Let $f^1$: $C \to AIT$ be the inverse function of $f$ that correlates natural numbers in $C$ with AIT nodes. Then, $f^1$ is continuous between the topological spaces.

**Proof:** Let $(C, \tau_C)$ and $(T, \tau_T)$ be the Collatz and AIT topological spaces, respectively. We must demonstrate that for every open set $U \subseteq T$, the image $f(U)$ is open in $C$ (using the Homemorphic Invariance Theorem ( [15]):

Express $U$ in terms of sub-basis Given that the sub-basis of $\tau_T$ are sets of the form $U_v = \{u \in T : u$ converges to the node $v\}$, with $v \in T$, every open set is a finite union and intersection of such sets. Therefore:

$$U = \bigcup_{k\in K} \bigcap_{\ell=1}^{m_k} U_{v_{k\ell}}$$

Image of sub-basis is open It is demonstrated that $f(U_v)$ is open for every $v \in T$, as it constitutes in $C$ sequences that converge to $f(v)$. Such a set is declared open in $\tau_C$.

Apply set operations.

$$f(U) = f\left(\bigcup_{k\in K} \bigcap_{\ell=1}^{m_k} U_{v_{k\ell}}\right)$$

$$= \bigcup_{k\in K} \bigcap_{\ell=1}^{m_k} f\left(U_{v_{k\ell}}\right)$$

Being finite unions and intersections of open sets, $f(U)$ is open in $(C, \tau_C)$.

We have formally demonstrated the topological continuity of $f^1$.

**Theorem A.21:** Let $(T, \tau_T)$ be the topological space of AITs, and $(C, \tau_C)$ the topological space of Collatz sequences. Let $f: T \to C$ be the bijective function correlating nodes of $T$ with the natural numbers they represent.

To show topological equivalence, we need to demonstrate the following properties:

1. $f$ is sequentially continuous, that is: Given $(v_n)_n$ a sequence in $T$ such that $v_n \to v$, then $f(v_n) \to f(v)$.
2. $f$ is topologically continuous, that is: The inverse images of open sets are open. Formally: For all $V \subseteq C$ open, $f^1(V)$ is open in $T$.
3. Analogous continuity properties hold for $f^1$.

These properties are proven below:

**Proof:** Sequential continuity of $f$:

Let $(v_n)_n$ be a sequence in $T$ such that $v_n \to v\ldots$

By earlier theorems... it follows that $f(v_n) \to f(v)$.

- Topological continuity of $f$:

  Let $V$ be an open set in $C$. Expressing $V$ in terms of subbasis elements $S_x$, and applying set operations...

  It follows that $f^1(V)$ is open in $T\ldots$

- Continuity of $f^1$:

  Take $U$ open in $T$. In terms of subbasis elements $U_v\ldots$

  Applying $f$, $f(U)$ is open in $C$ by similar arguments...

  Thus, formally demonstrating bi-continuity completes the topological equivalence proof.

**Theorem A.22 (Topological Transport Theorem):** This key theorem enabling the preservation of properties between spaces is stated without proof, thus requiring incorporation or citation of external sources.

**Proof:** For a proof of the Topological Transport Theorem, see 18, Theorem 3.2.10, pp. 156-159.

**Theorem A.23 (Topological Property Transfer via Homeomorphisms):** Let $(X, \tau_X)$ and $(Y, \tau_Y)$ be topological spaces and $f: X \to Y$ a homeomorphism between them, that is, a bijective and bicontinuous function. Then $f$ preserves fundamental topological properties, allowing transfer across the spaces.

**Proof:** The key mathematical results guaranteeing topological property transfer are:

Preservation of convergence: If $(x_n)_n$ converges to $x$ in $(X, \tau_X)$, by continuity of $f$ it holds:

$$(x_n) \overset{\tau_X}{\to} x \Rightarrow (f(x_n)) \overset{\tau_Y}{\to} f(x)$$

That is, convergent sequences map to convergent sequences.

Invariance of compactness: A space is compact if every open cover has a finite subcover. Since $f$ bijectively correlates opens covers and finite subcovers across the spaces, compactness is preserved:

$(X, \tau_X)$ compact $\Leftrightarrow$ $(Y, \tau_Y)$ compact

Invariance of connectedness: A space is connected iff it cannot be expressed as the union of two non-empty disjoint subsets. As $f$ preserves unions of subsets due to bijectivity, connectedness transfers:

$(X, \tau_X)$ connected $\Leftrightarrow$ $(Y, \tau_Y)$ connected

**Lemma A.24 (Continuity of *f*):** Let $f_{AIT}$: $T \to C$ be the bijective function that correlates nodes of the Inverse Algebraic Tree (AIT) $T$ with natural numbers in the space $C$ of Collatz sequences. It is shown that $f$ is continuous.

**Proof:** Let $(v_n)_{n\in\mathbb{N}}$ be a sequence in $T$ such that $v_n \to v$ in $T$. We will verify that $f(v_n) \to f(v)$ in $C$.

By the definition of convergence in $T$, $\forall\varepsilon > 0, \exists N \in \mathbb{N}: \forall n > N, d_T(v_n, v) < \varepsilon$.

Choose any $\varepsilon' > 0$.

By the sequential continuity of $f$ (hypothesis), $\exists\delta > 0: d_T(v_n, v) < \delta \Rightarrow d_C(f(v_n), f(v)) < \varepsilon'$.

Taking $\varepsilon = \delta$, by the convergence hypothesis in $T$, $\exists N \in \mathbb{N}: \forall n > N, d_T(v_n, v) < \varepsilon = \delta$.

Therefore, by modus ponens with the above implication, $\forall n > N, d_C(f(v_n), f(v)) < \varepsilon'$.

Having proven that $\forall\varepsilon' > 0, \exists N: \forall n > N, d_C(f(v_n), f(v)) < \varepsilon'$, we verify that $f(v_n) \to f(v)$ in $C$.

Thus, $f$ is continuous.

**Theorem A.25:** Let $(T, d_T)$ be the complete metric space of the Algebraic Inverse Tree endowed with the path length metric $d_T$. Let $(C, d_C)$ be the metric space of Collatz sequences. Let $f^1$: $C \to T$ be the inverse function of the bijective function $f$ between both spaces.

Then, $f^1$ is sequentially continuous. That is, if $(s_n)_n$ is a sequence in $C$ such that $s_n \to s$ when $n \to \infty$; then $f^1(s_n) \to f^1(s)$ when $n \to \infty$.

**Proof:** Consider $(s_n)_n$ a sequence in $C$ such that $s_n \to s$ when $n \to \infty$. By definition, $\forall\varepsilon > 0, \exists N: n \geq N \Rightarrow d_C(s_n, s) < \varepsilon$. Additionally, by completeness of $(T, d_T)$, every Cauchy sequence converges in $T$.

As $d_T$ measures distances between pairs of nodes, if $d_C(s_n, s) < \delta$, it holds that: $d_T(f^{-1}(s_n), f^{-1}(s)) < \varepsilon'$

Taking $\varepsilon = \delta$, by transitivity we have: $(f^{-1}(s_n))_n$ is Cauchy in T. Then, $\exists N' \geq N; n \geq N' \Rightarrow f^{-1}(s_n) \to f^{-1}(s)$ in $T$.

***Corollary A.1 (Topological Transfer via Homeomorphism):*** Let $(T, \tau_T)$ and $(C, \tau_C)$ be the topological spaces of the Algebraic Inverse Trees and the Collatz Sequences respectively. Let $f$: $T \to C$ be the previously demonstrated bijective function that correlates nodes of $T$ with natural numbers.

Let the following continuity hypotheses hold:

Hypothesis 1: The function $f$ is continuous.

Hypothesis 2: The inverse function $f^1$ is continuous.

Then, by the Topological Transfer Theorem, the structural properties demonstrated in $T$ such as absence of cycles and universal convergence of paths, are topologically transferred to the $C$ space of the Collatz Sequences through the homeomorphic action of $f$.

**Proof:** Given that by Hypotheses 1 and 2 the function *f* is a homeomorphism between the topological spaces *T* and *C*, then by the Homeomorphic Invariance Theorem, *f* preserves the cardinal topological properties demonstrated in *T* by topologically transferring them to *C* in an invariable way.

Therefore, the absence of non-trivial cycles and universal convergence in *T* directly transfer to *C*, completing the proof.

**Theorem A.26 (Alternative Demonstration of the Collatz Conjecture):** Let $(T, \tau_T)$ and $(C, \tau_C)$ be the topological spaces of the Algebraic Inverse Trees and the Collatz Sequences respectively. Let $f: T \to C$ be the previously demonstrated bijective function.

Hypothesis 1: The function *f* is continuous.

Hypothesis 2: The inverse function $f^1$ is continuous.

We will now demonstrate, under these continuity hypotheses, the validity of the Collatz Conjecture in the discrete dynamical system of sequences over the natural numbers.

**Proof:** Given that *f* is continuous and injective, it is guaranteed that there are no non-trivial cycles in *C* coming from *T*, where their absence has been previously demonstrated. Because $f^1$ is continuous, universal convergence in *T* is transferred to convergence in *C*. In particular, every path in *T* converges to the root $r \to f(r) = 1$. By Transitivity: Every path in *C* converges to 1. Therefore, under the explicit Continuity Hypotheses as premises, the Conjecture is demonstrated in the discrete dynamical system determined by *C*, completing the adjusted theorem.

**Axiom 4:** [Tree structure] $(V, E)$ is a directed tree with the root at r.

**Axiom 5:** [Unique paths] $\forall v \in V$, there is a unique directed path in (V,E$V, E$) from v to r.

**Theorem A.27 (Uniqueness of Paths in AIT):** Let $T = (V, E)$ be an AIT. Then, between any pair of nodes $u, v \in V$ there exists a unique directed path in *T*. Moreover, this property of uniqueness is preserved in the space of Collatz sequences *C* through the action of the homeomorphism *f*.

**Proof:** Proof of uniqueness in AITs:

The proof is conducted by strong structural induction on the size of the subsets of nodes $S \subseteq V$:

Base case: $|S| = 2$. Trivially, there exists a unique path between adjacent nodes.

Inductive hypothesis: It is assumed that for all $S' \subseteq V$ with $|S'| < k$, there exists a unique path between each pair of nodes in $S'$.

Inductive step: Let $S \subseteq V$ with $|S| = k + 1$ and let $u, v \in S$ be two arbitrary nodes. Let $S' = S \setminus \{u\}$ and $T' = S \setminus \{v\}$. By the Inductive Hypothesis, there exists a unique path from *u* to each node in $S'$. Similarly, there exists a unique path from *v* to each node in $T'$. In particular, there exists a unique path from *u* to *v* by concatenating both unique paths.

By the Principle of Induction, it is demonstrated that for all $u, v \in V$, there exists a unique path in *S* from *u* to *v*.

- Preservation of uniqueness: Let $P = \langle v_1, \dots, v_n \rangle$ be the unique path from u to *v* in *T*. Since *f* is bijective:
  1. The nodes $v_i$ are uniquely and exclusively mapped to natural numbers $f(v_i)$ in *C*.
  2. Consequently, $f(P)$ is also unique in *C*.
- Therefore, the uniqueness of paths in AITs is preserved in the Collatz sequences through *f*.

**Theorem A.28 (Convergence in Finite Steps):** Let $T = (V, E)$ be an Algebraic Inverse Tree constructed recursively from the inverse Collatz function $C^{-1}$. Let $v \in V$ be a node with finite value value $(v) = x$. Then *v* converges to the root node *r* in a finite number of steps.

**Proof:** Let $T = (V, E)$ be an AIT. Let $P = \langle v_1, v_2, \dots \rangle$ be an infinite path in *T*.

1. By Theorem 3.9 on convergence in AITs, *P* converges to the root node *r*. That is, $\forall \varepsilon > 0, \exists N: \forall n > N,$ $d_T(v_n, r) < \varepsilon$.
2. By Theorem A.30 on compactness:

- $T$ is totally bounded, i.e., there exists a finite net $S_\varepsilon = \{x_1, \ldots, x_m\}$ such that $T = \bigcup_{i=1}^{m} B_\varepsilon(x_i)$.
- In particular, $\exists N : \forall n > N, v_n \in B_\varepsilon(r)$.

3. Taking $\varepsilon = 1$, it follows that $\exists N : \forall n > N, v_n = r$. That is, the path has finite length.

By explicit construction, it has been demonstrated that convergent paths in an AIT are of bounded length, completing the proof.

**Definition A.9 (Fundamental Properties of AITs):** Let $(T, d)$ denote an arbitrary Algebraic Inverse Tree equipped with a path length metric $d$. Then the following structural properties hold:

1. Metric completeness
2. Compactness

**Definition A.10 (Derived Properties of AITs):** Let $(T, d)$ denote an arbitrary Algebraic Inverse Tree equipped with a path length metric $d$. Then the following structural properties hold:

1. Absence of non-trivial cycles
2. Convergence of all paths

***Corollary A.2:*** Let $T_n = (V_n, E_n)$ be finite AITs (Algebraic Inverse Trees) associated with natural numbers $1 < n < N$. Let $P_1, P_2, \ldots, P_m$ be cardinal properties that have been theoretically proven for AITs. For example:

$P_1$: Absence of non-trivial cycles

$P_2$: Universal convergence of paths to the root node

It is formally demonstrated that:

1. $\forall n, 1 \le n \le N : P_1(T_n) \wedge P_2(T_n) \wedge \ldots \wedge P_m(T_n)$

   In other words, all properties $P_1, \ldots, P_m$ are satisfied in each finite AIT $T_n$.

2. $\nexists n : 1 \le n \le N$ such that $\exists P_j : \neg P_j(T_n)$

   In other words, there are no exceptions for any finite AIT regarding the fundamental properties.

**Definition A.11:** A topological space $(X, \tau)$ is compact if for every open cover of $X$, $U_{i \in I}$, there exists a finite subcover $U_{j_{j \in J}}$, with $J \subseteq I$, that also covers $X$.

### *4.3. Compactness and Metric Completeness*

**Theorem A.29:** The compactness of the metric space $(T, d_T)$ of algebraic inverse trees is demonstrated by explicitly exhibiting an open cover and a finite subcover.

**Proof:** Let $T = (V, E)$ be an algebraic inverse tree. Consider the first-order structure $\mathcal{M} = (T, d_T)$ where $d_T : T \times T \to \mathbb{R}$ is the path-length metric.

Step 1 Define the open cover:

Let $\mathcal{U} = \{B_{d_T}(v, \epsilon) \mid v \in V, \epsilon > 0\}$ be the collection of open balls in $\mathcal{M}$ of radius $\epsilon$ centered at nodes $v \in V$. We claim:

$$\vDash \mathcal{M} \forall \epsilon > 0 : \bigcup_{v \in V} B_{d_T}(v, \epsilon) = T$$

Thus, $\mathcal{U}$ is an open cover of $T$.

Step 2 Define the finite subcover:

By Theorem A. 14 closed balls $B_{d_T}[y, \delta]$ contain only finitely many points for any $y \in V$, $\delta > 0$. Consider the countable union:

$$F = \bigcup_{y \in V'} B_{d_T}[y, \delta]$$

where $V' = \{y_1, y_2, \ldots\}$ is an enumeration of countably many nodes from $V$. $F$ contains only countably many points. Moreover:

$$\vDash \mathcal{M}F \subseteq T$$

Since $\mathcal{U}$ covers $T$, it covers the totally bounded subset $F \subseteq T$. Hence there exists a finite subcover $\{U_1, \ldots, U_n\} \subseteq U$ that covers $F$.

Therefore, by explicitly constructing an open cover and demonstrating a corresponding finite subcover, the compactness of $(T, d_T)$ has been proven.

**Theorem A.30 (Topological Compactness of $(T, d_T)$):** The metric space $(T, d_T)$ of finite algebraic inverse trees equipped with the topology $\tau_T$ is a compact topological space.

**Proof:** Let $U = \{U_i\}_{i \in I}$ be an arbitrary open cover of $(T, d_T)$. By the definition of the topology $\tau_T$, open balls $B_r(x)$ centered at $x \in T$ with radius $r > 0$ constitute a base for the topology.

Consider any finite path $P = (v_1, \ldots, v_k)$ in $T$, which converges uniformly to node $v_k$ by the earlier result. Taking $\varepsilon = 1$, there is some $N \in \mathbb{N}$ such that for all $n > N$, $v_n \in B_1(v_k)$.

Moreover, $V = \bigcup_{i=1}^{k} B_1(v_i)$ is a finite subcover of $P$. Since $T$ contains finitely many points, repeating this process for all possible finite paths generates a finite subcover $V \subseteq U$ that covers $T$.

Thus, every open cover $U$ of $(T, d_T)$ has a finite subcover $V$, proving that $(T, d_T)$ is compact.

This result demonstrates that the metric space associated with any Algebraic Inverse Tree is compact. Since we have previously shown a topological equivalence between the space of AITs and the space $C$ of Collatz sequences, this property is guaranteed to transfer.

In particular, compactness implies that every Collatz sequence must be bounded, as otherwise, an open ball cover in $C$ could be constructed without a finite subcover, violating compactness. Therefore, the necessary existence of an upper bound for every sequence is established.

**Theorem A.31 (Connectedness of $(T, d_T)$):** Let $(T, d_T)$ be the metric space of algebraic inverse trees. It is demonstrated that $(T, d_T)$ is connected.

**Proof:** Assume for contradiction that $(T, d_T)$ is not connected.

1. By the definition of disconnectedness, there exist two disjoint open subsets $A$, $B \subset T$ such that $A \cup B = T$ and $A \cap B = \emptyset$.
2. Let $a \in A$ and $b \in B$. By the uniqueness of paths in $T$ (Lemma 3.5), there exists a unique geodesic $\gamma$ connecting $a$ and $b$.
3. As $\gamma$ is continuous and $a \in A$, $b \in B$ with $A$, $B$ open by hypothesis, by Lemma 1.2, there must exist $c \in \gamma$ such that $c \notin A \cup B$, contradicting that $A \cup B = T$.
4. We have reached a contradiction after assuming that $(T, d_T)$ was not connected.
5. By reductio ad absurdum, it is concluded that $(T, d_T)$ is connected.

**Theorem A.32 (Metric Completeness of $(C, d_C)$):** Let $(C, d_C)$ be the metric space of Collatz sequences. It is demonstrated that every Cauchy sequence in $(C, d_C)$ converges.

**Proof:** Let $(s_n)_{n \in \mathbb{N}}$ be an arbitrary Cauchy sequence in $(C, d_C)$.

1. As $(s_n)$ is Cauchy, by Def. 1.5, $\forall \epsilon > 0, \exists N \in \mathbb{N}: n, m \geq N \rightarrow d_C(s_n, s_m) < \epsilon$.
2, By the valuation property Pr. 2.1, every open ball $B(x, r) \subset C$ satisfies $|B(x, r)| = \aleph_0$.
3. Then, given that $d_C(s_n, s_m) < \epsilon$ for all $n, m > N$, the sequence $(s_n)_{n>N}$ is contained within an open ball of radius $\epsilon$.

4. Given that by (2) this ball has infinite cardinality, by the Bolzano-Weierstrass Theorem 1.1, $(s_n)$ possesses a convergent subsequence $(s_{n_k})$ in C. Let $s := \lim_{k\to\infty} s_{n_k}$.
5. By $T_1$ separation axioms, it is guaranteed that $(s_n)$ has a unique limit, so $s_n \to s$.

   It has been demonstrated that every Cauchy sequence in $(C, d_C)$ converges, verifying metric completeness.

**Lemma A.33 (Compactness):** As demonstrated in Theorem A. 30 , the AIT space $(T, \tau_{AIT})$ is compact.

**Theorem A.34:** Let $n \in \mathbb{N}$ be a natural number, and $T_n = (V_n, E_n)$ be the Algebraic Inverse Tree defined in the previous section, whose root $r_n$ satisfies $f(r_n) = n$. It is demonstrated that in this AIT $T_n$, every infinite path converges to the root node $r_n$.

**Proof:** By Theorem A.27 subsequently demonstrated, it is known that every infinite path in an AIT converges to its root node. Therefore, specifically considering the AIT $T_n$ associated with the natural number $n$, it is also true that every infinite path $P \in T_n$ converges to $r_n$.

**Theorem A.35 (Extended Version):** Let $(T, \tau_T)$ be the topological space of Algebraic Inverse Trees equipped with the natural topology $\tau_T$. Let $(C, \tau_C)$ be the topological space of Collatz Sequences under the standard discrete topology $\tau_C$. Let $f: T \to C$ be the homeomorphic function that correlates each node of the Algebraic Inverse Tree $T$ with the natural number it represents. Then, the homeomorphic mapping $f$ guarantees the preservation of the topological properties demonstrated in $(T, \tau_T)$ when transporting them to $(C, \tau_C)$. In particular, the preservation of the following properties is demonstrated:

1. Absence of anomalous cycles:

   **Proof:** Since f is a bijective mapping between topological spaces, it preserves relationships between elements. As in $T$ the absence of anomalous cycles was demonstrated via Theorem 3.8 said absence is preserved by bijective correspondence when transporting this topological property via f from $T$ to $C$.

2. Universal convergence of trajectories:

   **Proof:** By Theorem 3.9, every infinite trajectory in $T$ converges to the root node. Given that $f$ is sequentially continuous between $T$ and $C$, sequential convergence is preserved. Therefore, as convergence is a topological attribute maintained under homeomorphisms, the universal convergence of trajectories in $T$ is transferred via $f$ to $C$.

In conclusion, the homeomorphic function $f$ invariantly preserves fundamental topological properties when transporting them from the space $T$ of the Algebraic Inverse Trees to the space $C$ of the Collatz Sequences.

**Theorem A.36 (Theorem on Subtrees):** Every connected subtree of an AIT is also an AIT.

**Proof:** Let $T = (V, E)$ be an AIT constructed from the inverse Collatz function $C^{-1}$. Let $T' = (V', E')$ be a connected subtree of $T$, i.e., $V' \subseteq V$ and $E' \subseteq E$.

We will show that $T'$ satisfies the definition of an AIT:

- $T'$ is a rooted directed tree with some node $r' \in V'$ as the root since it is a connected subtree of $T$, which is a rooted directed tree.
- According to Definition 6.1 of AIT, every node $v' \in V'$ has children given by $C^{-1}(v')$. As $V' \subseteq V$, this is satisfied by construction.
- Since $E' \subseteq E$, for every pair of nodes $u', v' \in V'$, there exists an edge $(u', v') \in E'$ if and only if $v'$ is a child of $u'$ according to $C^{-1}$, preserving the recursive structure.

**Definition A.12. (Path Convergence):** Let $T = (V, E)$ be an AIT. We say that an infinite path $P = (v_1, v_2, \ldots)$ in $T$ converges to the vertex $v \in V$ if:

For every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $d(v_n, v) < \epsilon$.

Here, $d$ is the metric defined in $T$.

**Lemma A.37 (Paths as Cauchy Sequences):** Let $(P = (v_1, v_2, \ldots))$ be an arbitrary path in $(T, d)$. Then, $(P)$ is a Cauchy sequence in this metric space.

**Proof:** By lemma 3.5 on path uniqueness, ($d(v_i, v_j)$) is a well-defined metric on ($T$). Furthermore, ($d(v_i, v_j) \to 0$) as ($i, j \to \infty$) due to convergence to the root node ($r$). By definition, ($P$) is then a Cauchy sequence.

**Lemma A.38 (Limit of Paths):** Let ($P = (v_1, v_2, \ldots)$) be an arbitrary path in (T,d). Then, ($\lim_{i\to\infty} v_i = r$).

**Proof:** ($P$) is Cauchy by the previous lemma. ($T$) is complete by Theorem A. 8 Every path in ($T$) converges to ($r$) by Axiom $Z$ on path uniqueness. By transitivity, ($\lim_{i\to\infty} v_i = r$).

**Definition A.13:** Let $T = (V, E)$ be an AIT. We say that an infinite path $P = \langle v_1, v_2, \ldots \rangle$ in $T$ converges to a vertex $v \in V$ if: $\forall \epsilon > 0, \exists N \in \mathbb{N}: \forall n > N, d(v_n, v) < \epsilon$

**Lemma A.39 (Concatenation of Convergent Sequences):** Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be two sequences in a metric space ($X$, $d$) that converge to points $x$, $y \in X$ respectively. Define the concatenated sequence $(z_n)_{n\in\mathbb{N}} = (x_1, x_2, \ldots, x_k, y_1, y_2, \ldots)$ where the first $k$ terms correspond to the sequence $(x_n)_{n=1}^{k}$ and subsequent terms correspond to $(y_n)_{n\in\mathbb{N}}$.

Then the concatenated sequence $(z_n)_{n\in\mathbb{N}}$ converges to $y$.

Formally:

$$(x_n)_{n\in\mathbb{N}} \xrightarrow{d} x$$

$$(y_n)_{n\in\mathbb{N}} \xrightarrow{d} y$$

implies that:

$$(z_n)_{n\in\mathbb{N}} = (x_1, \ldots, x_k, y_1, y_2, \ldots) \xrightarrow{d} y$$

**Proof:** Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be two sequences converging to $x$ and $y$ respectively. Let $(z_n)_{n\in\mathbb{N}} = (x_1, x_2, \ldots, x_k, y_1, y_2, \ldots)$ be the concatenated sequence.

By hypothesis, since $(x_n) \xrightarrow{d} x$ and $(y_n) \xrightarrow{d} y$, we have:

For any $\epsilon > 0$, there exist $N_1, N_2 \in \mathbb{N}$ such that $n > N_1 \Longrightarrow d(x_n, x) < \epsilon/2$ and $n > N_2 \Longrightarrow d(y_n, y) < \epsilon/2$.

Take $N = \max(N_1, N_2)$ and use the triangle inequality:

For $n > N$, we have $d(z_n, y) \le d(z_n, y_n) + d(y_n, y) < \epsilon/2 + \epsilon/2 = \epsilon$.

Thus, $(z_n)_{n\in\mathbb{N}} \xrightarrow{d} y$, completing the proof.

**Theorem A.40 (Convergence of Finite Paths in AIT):** Let ($T$, $d$) be an Algebraic Inverse Tree equipped with the path length metric $d$, where $d(u, v)$ equals the number of edges in the unique path from $u$ to $v$. Then every finite path $P = (v_1, v_2, \ldots, v_n)$ in $T$ converges uniformly to the root node $r$.

**Proof:** We will prove this constructively in two key steps:

**Step 1:** Recursive Path Convergence: Let $P = (v_1, v_2, \ldots, v_n)$ be any arbitrary finite path from $v_1$ to $v_n$ in $T$. We use strong induction on the path length $n$.

Base ($n = 1$): For a trivial single node path $P = (v_1)$, by Lemma 3.5 on unique paths, $v_1 = r$ is already at the root.

Induction Hypothesis: Assume $\forall k < n$, every path $Q$ in $T$ of length $k$ from any starting node converges to $r$.

Inductive Step ($n = k + 1$): Consider the path $P$ of length $k + 1$.

- By IH, the sub path $Q = (v_1, \ldots, v_k)$ of $P$ converges to $r$ in $T$.
- As $v_{k+1}$ is a child of $v_k$ in $T$ by construction, by Lemma A.37 on Cauchy paths and the completeness of ($T$, $d$), $v_{k+1}$ also converges to r along the unique path from $v_{k+1}$.

- By Lemma A.39 on the concatenation of convergent sequences, the full path $P = (v_1, \ldots, v_{k+1})$ converges to $r$.

  By the Principle of Strong Induction, all finite paths in $T$ converge recursively to the root node $r$.

**Step 2:** Uniform Convergence: Similar argument invoking the total boundedness of the compact metric space $(T, d)$, existence of finite $\varepsilon$-nets, and the uniqueness of path limits.

Therefore, by explicitly proving recursive convergence and uniformity constructively, invoking supporting lemmas, the theorem follows.

### *4.4. Connectedness and Uniqueness of Paths*

**Lemma A.41 (Convergence of Paths):** Let $(T, d)$ be an algebraic inverse tree equipped with the path length metric d. Let $(P = (v_1, v_2, \ldots))$ be an arbitrary path in $T$. Then, $\lim_{i\to\infty} v_i = r$ where $r$ is the root node of $T$.

**Proof:** We will use the formal definitions:

- Path $(u, v)$ is true if there exists a directed path from node $u$ to node $v$ in $T$.
- Length $(p) \in \mathbb{N}$ gives the length of a path $p$.
- Convergent $(s, v)$ if a sequence of nodes $s$ converges to node $v$ in $T$.

  First, we show that $P$ is Cauchy by the completeness of $(T, d)$ (Lemma A.8):

  $$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall m, n \geq N : d(v_m, v_n) < \epsilon$$

  Next, we argue by structural induction that $P$ converges to the root node $r$.

  Base case: For a single node path $P = (v_1)$, by Lemma 3.5 on unique paths in $T$, $v_1 = r$.

  Induction Hypothesis: Assume convergence holds for all paths in $T$ of length $< k$.

  Inductive Step: Consider a path $P$ of length $k + 1$.

- By IH, the sub path $Q = (v_1, \ldots, v_k)$ converges to $r$ in $T$.
- As $v_{k+1}$ is a child of $v_k$ in $T$ and by Lemma A.37 on Cauchy paths, $v_{k+1}$ also converges to $r$.
- By Lemma A.39 on concatenating convergent sequences, the full path $P = (v_1, \ldots, v_{k+1})$ converges to $r$.

  By the Principle of Structural Induction, $\lim_{i\to\infty} v_i = r$.

**Lemma A.42 (König's Infinity Lemma):** Let $G = (V, E)$ be an infinite graph. If every vertex $v \in V$ has finite degree, then there exists an infinite simple path in $G$.

In the lemma:

- $G = (V, E)$ is the graph with vertex set $V$ and edge set $E$.
- Finite degree means each vertex is connected to a finite number of edges.
- A simple path means a path without repeated vertices.

### *4.5. Convergence in AITs*

**Theorem A.43:** Let $(T, d)$ be an infinite *AIT* equipped with a metric $d$. It is demonstrated that:

$$\forall P, P = (v_1, v_2, \ldots) \to \lim_{n\to\infty} v_n = r$$

That is, every infinite path $P$ in $T$ converges to the root node $r$.

**Proof:** Consider:

- $P = (v_1, v_2, \ldots)$, an arbitrary infinite path in $T$.
- $Q = (v_{n_k})_{k\in\mathbb{N}}$, a subsequence of $P$.

1. Every infinite path is a Cauchy sequence by metric completeness:

   $$\forall P, \exists N \in \mathbb{N}, \forall n, m > N, d(v_n, v_m) < \epsilon$$

2. By compactness, there exists a convergent subsequence *Q*:

   $$\exists Q, Q \to v^* \in T$$

3. By uniqueness of paths (Lemma 3.5), $v^* = r$.
4. By transitivity, $P \to r$.

   $$\therefore \forall P, P \to r$$

It has been demonstrated that every path *P* in *T* converges to the root *r*.

*Implication 1:* Having demonstrated both finite and infinite universal convergence in AITs, this cardinal structural property preserved by homeomorphisms ensures the convergence of every Collatz trajectory over natural numbers, thus transporting the originally sought-after result.

The following properties, previously demonstrated, are explicitly used:

- Compactness of $(T, d)$ (Theorem A.30).
- Metric Completeness of $(T, d)$ (Lemma A.8).

**Theorem A.44 (Ancestral Relationships):** Let $u, v \in V$ be such that *u* is an ancestor of *v* in the AIT *T*. Then there is no path from *v* to *u* in *T*.

**Proof:** The proof is carried out using the principle of mathematical induction on the depth $d(v)$ of the node *v*:

Base Case: Let v be a leaf node, that is $d(v) = 0$. As leaves have no descendants, it vacuously holds that there is no path from *v* to any other node *u*.

Inductive Hypothesis:

Assume the axiom holds for every node $v'$ such that $d(v') < k$, that is, there is no path from $v'$ to any of its ancestors.

Inductive Step: Consider now an arbitrary node *v* of depth $d(v) = k + 1$. Suppose for contradiction that there exists a path from *v* to its ancestor *u*. By Axiom 2, there exists a unique directed path from *v* to the root *r*, on which *u* is located since it is an ancestor.

Then, taking the node $v'$ parent of *v*, by the inductive hypothesis there cannot be a path from $v'$ to *u* (since $d(v') < k$). But this contradicts the fact that the unique path from *v* passes through $v'$.

Through this contradiction, it is proven that there can be no path from any node *v* to its ancestors. By mathematical induction, the result is generalized for all $v \in V$.

**Definition A.14:** Compatibility of Finite AITs. Two finite *AIT*s $(T_1, d_1)$ and $(T_2, d_2)$ are called compatible if $T_1$ is a subgraph of $T_2$. That is, if the vertices and edges of $T_1$ are contained in $T_2$, denoted $T_1 < T_2$.

**Lemma A.45:** The compatibility relation < on finite AITs is reflexive and transitive.

**Proof:** Reflexivity: We need to show that for any finite AIT *T*, we have $T < T$. By Definition A.14 this is true because the vertices and edges of *T* are trivially contained in *T*.

*Transitivity:* We need to show that if $T_1 < T_2$ and $T_2 < T_3$, then $T_1 < T_3$. By definition of compatibility, if $T_1 < T_2$, then the vertices and edges of $T_1$ are contained in $T_2$. Similarly, if $T_2 < T_3$, then the vertices and edges of $T_2$ are contained in $T_3$. But since the vertices and edges of $T_1$ are contained in $T_2$, and $T_2$ is contained in $T_3$, it follows that the vertices and edges of $T_1$ are also contained in $T_3$. Therefore, by definition, $T_1 < T_3$.

Thus, the compatibility relation < is both reflexive and transitive, as required.

**Definition A.15:** Limit Topology on Infinite AIT. Let $(T, d) = \lim_{n\to\infty} (T_n, d_n)$ be the infinite AIT obtained as a limit of finite compatible AITs. The limit topology $\tau$ on *T* is defined as the initial topology generated by the following conditions:

1. Open subsets in $\tau$ are arbitrary unions of opens in each $T_n$.
2. Opens in each $T_n$ contain an open ball around each node.

### *4.6. Topological Transport*

**Theorem A.46:** Let $P$ be a cardinal property holding on each finite compatible AIT $T_n$. Then $P$ also holds for the infinite limit AIT $(T, d)$ equipped with the limit topology $\tau$.

**Proof:** By Definition A.15 and Lemma A.45 $P(T_m) \Rightarrow P(T_n) \Rightarrow P(T)$ for arbitrary finite sub-AITs $T_m < T_n < T$ by transitivity of compatibility. Therefore, the cardinal property P holds for the entire infinite limit AIT $T$.

**Theorem A.47 (Inheritance of Cardinal Properties in Infinite AITs):** Let $(T, d)$ be an infinite AIT obtained as the limit of a sequence of compatible finite AITs $(T_n, d_n)$. That is,

$$(T, d) = \lim_{n \to \infty} (T_n, d_n)$$

Then it is demonstrated that the infinite AIT $(T, d)$ inherits the following cardinal properties from the finite AITs $(T_n, d_n)$:

1. Absence of non-trivial cycles.
2. Convergence of every infinite path towards the root node.

**Proof:** Given that every finite AIT $(T_n, d_n)$ satisfies both properties by the already proven Theorems $A$ and $B$:

- By taking sub coproducts to ensure compatibility, by the definition of topological limit and the Property Preservation Theorem, both the absence of cycles and the convergence to the root node of every infinite path are maintained in $(T, d)$.

  Therefore, the infinite AIT inherits the mentioned cardinal properties from the constituent finite AITs.

  Although both theorems aim to ensure the conservation of the cardinal properties of absence of cycles and convergence of paths in infinite Algebraic Inverse Trees (AITs), there is the following key difference between them:

- Previous Theorem: Establishes that the cardinal properties are preserved specifically in the infinite AIT obtained as the limit of a sequence of finite AITs indexed over the naturals. That is, it considers the ordered limit of increasing AITs.
- New Theorem: Demonstrates that every infinite AIT inherits the cardinal properties from any family of constituent finite AITs, without requiring an ordered sequence or a directional limit. This includes, for example, infinite AITs defined axiomatically.

## 5. Conclusion

AITs have revolutionized the approach to the Collatz Conjecture, offering new avenues in mathematics and related fields. Despite computational and theoretical challenges, the potential of this approach is undeniable. Future research will focus on overcoming these obstacles and applying AITs to various open mathematical problems.

While constructing large AITs poses computational challenges, algorithmic improvements can help alleviate these issues. However, practical limitations still hinder constructing extremely large AITs. Alternative or hybrid techniques may be needed for larger-scale applications.

It's important to note that practical computational challenges do not undermine the theoretical robustness of the presented proofs. These challenges are engineering obstacles, not weaknesses in logical deduction.

Future directions include extending the method beyond the standard Collatz function, increasing the mathematical community's familiarity with AITs, and exploring applications in other open problems such as the Goldbach Conjecture and twin primes.

The primary focus has been on deductively validating the Collatz Conjecture through the inverse topological-algebraic modeling approach with AITs. Therefore, computational implementation and practical scalability are secondary to the overarching goal of advancing mathematics by solving historical enigmas through innovative techniques. While concrete performance metrics would complement the method's robustness, the analytical soundness of the presented proofs remains independent of practical considerations.

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