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A Unified Statistical Relativistic Theory of The Generalized Brownian Motion Manifold

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Abstract

The current letter complements my info-geometric discoveries, especially those provided from two papers of mine. Fundamentally, the statistical relativization of the Generalized Brownian Motion Manifold. Following this innovative and unprecedented track of research will open a plethora of numerous info-geometric investigations to many unexplored related phenomena in the hope to uncover more new interpretations for the Generalized Brownian Motion Manifold from an info-geometric perspective.

Keywords: *Einsteinian relativities, The generalized brownian motion, Information geometry*

The reader is advised to consult (Mageed *et al.*, 2022; Mageed, 2024) as this letter is a continued track of research for both papers. Potentially, for the introduction and the definitions, the reader can consult (Mageed *et al.*, 2022; Mageed, 2024; Mageed and Kouvatsos, 2019, 2021; Mageed *et al.*, 2023(a); Mageed *et al.*, 2023(b); Mageed and Zhang, 2022; Minyoung *et al.*, 2022; Parr *et al.*, 2020; Ito and Dechant, 2020; Di Giulio and Tonni, 2020; Barbaresco, 2021; Thiruthummal and Kim, 2022; Ito, 2023; Li, 2022).

1. Introduction

1.1 Information Geometry

The fundamental idea of IG is to apply non-Euclidean geometry tools and techniques to probability theories and stochastic processes. A manifold is an infinite-dimensional manifold contained in a topological finite-dimensional Cartesian space, $\mathbb{R}^n$. All that is needed to characterise $\mathbb{R}^n$ is topological space, which can be defined as a collection of points and their respective neighbourhoods that meet governing axioms for neighbourhoods-points connectors. Furthermore, IG facilitates the intuitive thinking behind the SMs' description. It should be noted that while figures can be displayed in coordinate charts and other visual aids, they should be understood as strictly abstract geometric figures. One might recognise the enormous significance of IG at a deeper level, as shown by Figure 1.

To our knowledge, the current paper is the first ever to revolutionize classic Brownian Motion Theory (BMT) by devising the Info-Geometric analysis of (GBM).

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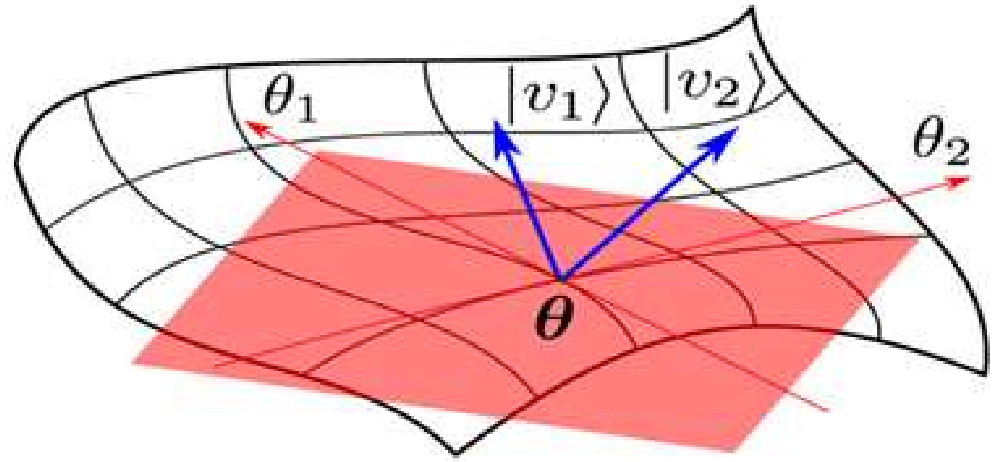

Figure 1: SM's Parametrization

In the context of this paper, Ricci curvature measures the deviation of the Riemannian metric (RM) from the standard Euclidean metric (EM) and how scalar curvature measures the deviation in the volume of a geodesic ball from the volume of a Euclidean ball of the same radius (c.f., Figure 2).

Geodesics are the analogue of straight lines in Euclidean space and possess many of the same properties as straight lines. In IG, the Fisher information metric (FIM) measures closeness of the shape between two distribution functions, it is also proportional to the amount of information that the distribution function contains about the parameter of the probability density function of the SM.

1.2. Generalized Brownian Motion (GB)

The Tsallis entropy reads:

$$\frac{1}{(q-1)}\left(1-\sum_{n=1}^{\infty}(p(n))^q\right) \quad ...(1.1)$$

where $p(n)$ is the probability of being in the n^{th} state. As $q \to 1$, H_q reduces to the Boltzmann-Gibbs entropy.

In the continuous domain, H_q reads:

$$H_q(X) = \frac{1}{q-1}\left(1-\int_{-\infty}^{\infty}[f_X(x)]^q\ dx\right) \quad ...(1.2)$$

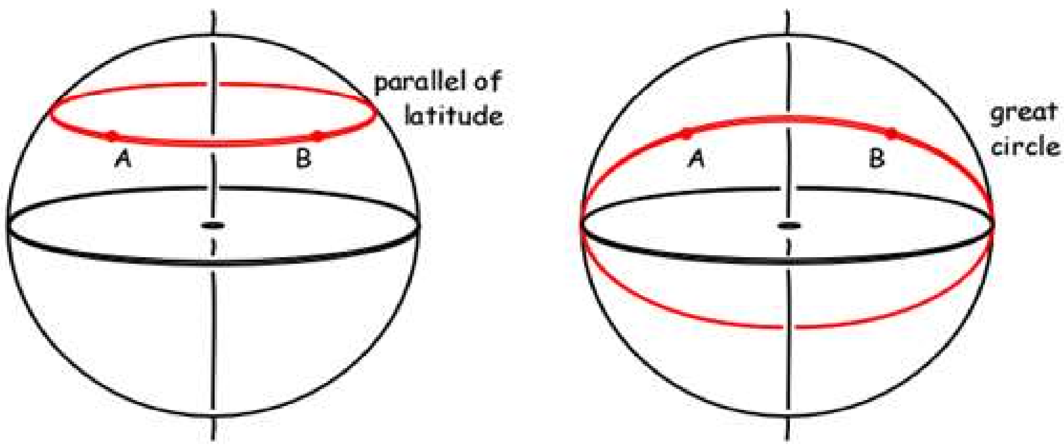

Figure 2: Geometric Representation of Geodesics on Curved Surfaces

where $f_X(x)$ is the probability density function (PDF) of X.

A q-Gaussian PDF reads:

$$f(x) = \frac{\sqrt{\beta}}{C_q} e_q(-\beta x^2) \quad ...(1.3)$$

where,

$$e_q(x) = [1 + (1-q)x]^{1/(1-q)} \quad ...(1.4)$$

is called the q-exponential, C_q is a normalization constant, and $\beta > 0$ is a scale parameter. In the range of extensive values of the information theoretic parameter q, $1 < q < 3$, the q-Gaussian distribution is a rescaled version of the Student's t-distribution with $\nu = \frac{3-q}{q-1}$ degrees of freedom. The scaling is such that the distributions are the same if $\beta = \frac{\nu+1}{2\nu} = \frac{1}{3-q}$. It is notable to state that the extensive values assigned to the information theoretic parameter q justifies the physical interpretation of Brownian Motion.

1.3 Random Diffusivity

Consider the stochastic differential equation:

$$dX(t) = \nu dt + \sqrt{D} dB(t) \quad ...(1.5)$$

where $B(t)$ is a Brownian motion, and D is a random variable that is independent of $B(t)$. Here the stochastic differential equation is regarded as being conditioned on D. If the probability density function, $f_D(x)$, of D is given by:

$$f_D(x) = \delta(x - D_0) \quad ...(1.6)$$

then D is a constant, and the distribution of the displacement due to diffusion, $X(t) - X(0) - \nu t$, is a Gaussian (note that the Gaussian distribution maximizes the Boltzmann-Gibbs entropy). This naturally leads to the question of whether there are distributions of D that would make the distribution of $X(t) - X(0) - \nu t$ maximize the Tsallis entropy. We will answer this question in the affirmative and explicitly construct the appropriate distribution for D.

Suppose that:

$$D \sim D_0 \left({}^{\nu}/_{V}\right)^2 \equiv g(V) \quad ...(1.7)$$

where $V \sim \chi^2(\nu)$ is a chi-squared distribution with ν degrees of freedom and ~ denotes that two random variables have the same distribution. Then the distribution of $X(t) - X(0) - \nu t$ takes the form

$$X(0) - \nu t \sim \sqrt{Dt} Z \quad ...(1.8)$$

$$\sim \sqrt{D_0 t} \frac{Z}{{}^{V}/_{\nu}} \quad ...(1.9)$$

where Z is a standard normal random variable.

Equation (1.7), implies

$$f_D(x) = f_V(g^{-1}(x)) \left| \frac{d}{dx} g^{-1}(x) \right|$$

$$= \frac{D_0^{(\frac{\nu}{4})} \nu^{(\frac{\nu}{2})} x^{-(\frac{\nu}{4}+1)}}{2^{(\frac{\nu}{2}+1)} \Gamma(\frac{\nu}{2})} e^{(-\frac{\nu\sqrt{D_0}}{2\sqrt{x}})} \quad ...(1.10)$$

Figure 3 shows several plots of $f_D(x)$.

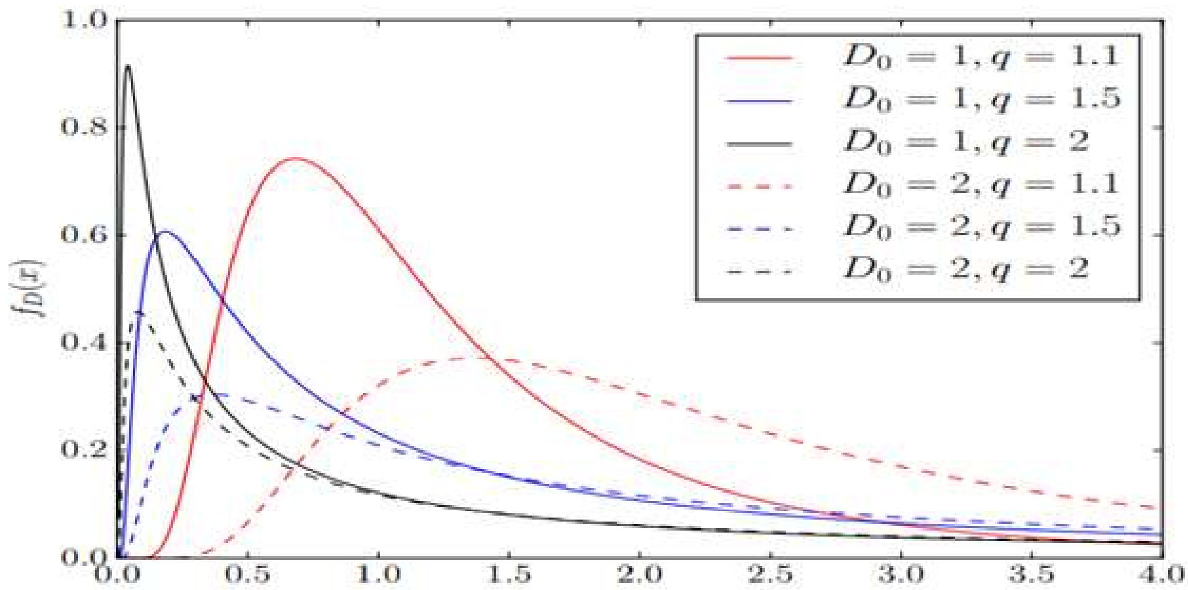

Figure 3: Several Plots of $f_D(x)$ for Several Combinations of q and D_0

Brownian motion is an idealized approximation to actual random dynamics that has been extensively investigated over a long period time, but possibly still not thoroughly understood. Figure 4 shows a numerical simulation of paths (bundle) from point a to point b for particles in a constant force field such as weight.

This current study contributes to:

- The provision of the Ricci Tensor of GBM manifold
- Revealing Ricci scalar of GBM manifold
- Obtaining Einstein and Stress Energy tensors which unifies GBM significantly with both general and special relativity.

The rest of this paper is organised as follows: Section 1 lays out a brief introduction to Information geometry, IG and Generalized Brownian Motion , GBM. Important IG definitions are provided by section 2. In section 3, Ricci scalar, $\mathcal{R}$

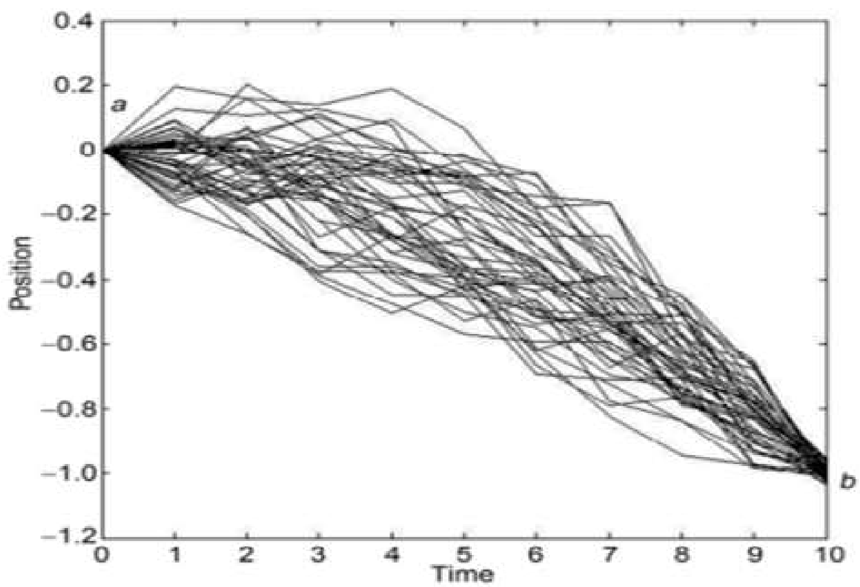

Figure 4: Numerical Simulation of Paths from Point a to Point b for Particles in a Constant Force Field Such as Weight

and the α-curvature Riemannian Tensor, Π-sectional curvature tensor, $\beth$ are calculated. In section 4, the Ricci Tensor, Υ, the curvature of space time (einestein tensor) $\wp$ and stress energy tensor, Ω are calculated. Concluded remarks and future work are given in section 5.

The following pivotal definitions are taken from (Mageed *et al.*, 2022; Mageed, 2024; Mageed and Kouvatsos, 2019, 2021; Mageed *et al.*, 2023(a); Mageed *et al.*, 2023(b); Mageed and Zhang, 2022; Mageed, 2023; Minyoung *et al.*, 2022; Parr *et al.*, 2020; Ito and Dechant, 2020; Di Giulio and Tonni, 2020; Barbaresco, 2021; Thiruthummal and Kim, 2022; Ito, 2023; Li, 2022).

2. Definitions

2.1. Definition

plz chk

1. Under the θ coordinate system, the α-curvature RiemannianTensors, $R_{ijkl}^{(\alpha)}$ are defined by

$$R_{ijkl}^{(\alpha)} = \left[\left(\partial_j \Gamma_{ik}^{s(\alpha)} - \partial_i \Gamma_{jk}^{s(\alpha)}\right) g_{sl} + \left(\Gamma_{j\beta,l}^{(\alpha)} \Gamma_{ik}^{\beta(\alpha)} - \Gamma_{i\beta,l}^{(\alpha)} \Gamma_{jk}^{\beta(\alpha)}\right)\right], i,j,k,l,s,\beta = 1,2,3,\dots,n \quad ...(2.1)$$

where $\Gamma_{ij}^{k(\alpha)} = \Gamma_{ij,s}^{(\alpha)} g^{sk}$

2. The α-Ricci curvatures (Ricci Tensors), $R_{ik}^{(\alpha)}$ read:

$$R_{ik}^{(\alpha)} = R_{ijkl}^{(\alpha)} g^{jl} \quad ...(2.2)$$

3. The α-sectional curvatures, $K_{ijij}^{(\alpha)}$ reads~:

$$K_{ijij}^{(\alpha)} = \frac{R_{ijij}^{(\alpha)}}{(g_{ii})(g_{jj}) - (g_{ij})^2}, i,j = 1,2,\dots,n \quad ...(2.3)$$

$K_{1212}^{(\alpha)} = K^{(\alpha)}$ is called α-Gaussian curvature reads:

$$K^{(\alpha)} = \frac{R_{1212}^{(\alpha)}}{det(g_{ij})} \quad ...(2.4)$$

4. The Ricci curvature Tensor (RCT) is simply a contraction of the Riemannian Tensor.

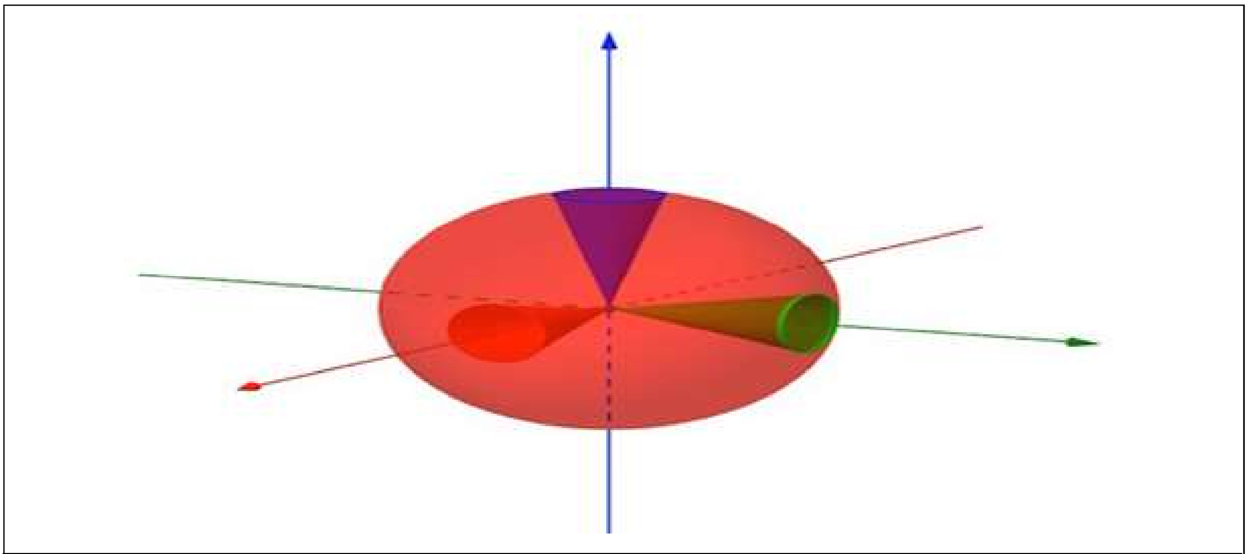

Figure 5: (RCT) Describes How Conical Regions in the Manifold Differ in Volume from the Equivalent Conical Regions in Euclidean Space

2.2. Scalar Curvature (Ricci Scalar), $\mathcal{R}$ and Einestein Tensor, ℘

The scalar curvature (Ricci Scalar), $\mathcal{R}$ is the contraction of Ricci Tensor.

$$\mathcal{R} = R_{ij}^{(\alpha)} g^{ij}, i, j = 1,2,3, \ldots \quad ...(2.5)$$

$$\mathcal{R} = 2K_G \quad ...(2.6)$$

K_G defines the Gaussian Curvature and $\mathcal{R}$ is two-dimensional

The Ricci scalar $\mathcal{R}$ has a meaning very similar to the Gaussian Curvature. If we imagine instead of taking a circle, taking a generalized n-1 sphere, i.e., the set of all points a geodesic distance ϵ from a given starting point x_0^{ζ}. We can calculate the area of this sphere in flat space, but in a curved space the area will deviate from the one we calculated by an amount proportional to the curvature. Thus, the Ricci Scalar is:

$$\mathcal{R} = lim_{\epsilon \to 0} \frac{6n}{\epsilon^2}[1 - \frac{A_{curved}\ (\epsilon)}{A_{flat}\ (\epsilon)}] \quad ...(2.7)$$

Ricci scalar completely captures the curvature of the surface.

Notably,

$$G_{ij} = R_{ij}^{(\alpha)} - \frac{\mathcal{R}}{2} g_{ij} = \frac{8\pi \mathcal{g} \varpi_{ij}}{c_{light}^4} \quad ...(2.8)$$

where G_{ij} is the Curvature of **Spacetime** (Einstein tensor), ℘, $R_{ij}^{(\alpha)}$ is the Ricci tensor of the **spacetime** represented by the metric g_{ij}, $\mathcal{R} = R_{ij}^{(\alpha)} g^{ij}, i, j = 1,2,3, \ldots, i$, is the Ricci scalar or scalar curvature, $\mathcal{g}$ is the universal gravitational constant, c_{light} is the speed of light, and ϖ_{ij} are the components of the stress-energy tensor, ϖ, describing generically the matter-energy distributions in the **spacetime**.

3. RICCI Scalar, $\mathcal{R}$, α-Curvature Rienmmanian Tensor, Π and α-Sectional Curvature Tensor, ℶ

3.1. Scalar Curvature (Ricci Scalar), $\mathcal{R}$

Theorem 1.1: The Ricci Scalar, $\mathcal{R}$ of the GBM Manifold is given by:

$$\mathcal{R} = \frac{4\left(\nu\psi_1\left(\frac{\nu}{2}\right) - 4\right)}{\nu^2\left((ln2 - 3) + \psi\left(\frac{\nu}{2}\right)\right)} \quad ...(3.1)$$

ψ, ψ_1 are the digamma and trigamma functions, respectively (Mageed, 2024)

Proof

The Gaussian Curvature K_G of GBM Manifold (Mageed, 2024) is given by:

$$K_G = \frac{2(\nu\psi_1\left(\frac{\nu}{2}\right) - 4)}{\nu^2((ln2 - 3) + \psi(\frac{\nu}{2}))} \quad ...(3.2)$$

Following (2.6), it holds that:

$$\mathcal{R} = 2K_G = \frac{4(\nu\psi_1\left(\frac{\nu}{2}\right) - 4)}{\nu^2((ln2 - 3) + \psi(\frac{\nu}{2}))} \text{ ...(c.f., (3.1))}$$

This proves our theorem.

3.2. The α-Curvature Rienmmanian Tensor, ב

In this section, α-curvature rienmmanian tensor, Π is obtained. This unifies GBM manifold with Riemannian geometry, which will be used to devise the α-sectional curvatures tensor, ב.

Theorem 3.2: The α-Curvature Rienmmanian Tensor, Π of the GBM Manifold is given by:

$$\Pi = \begin{pmatrix} R_{1111}^{(\alpha)} & R_{1112}^{(\alpha)} & R_{1121}^{(\alpha)} & R_{1122}^{(\alpha)} \\ R_{1211}^{(\alpha)} & R_{1212}^{(\alpha)} & R_{1221}^{(\alpha)} & R_{1222}^{(\alpha)} \\ R_{2111}^{(\alpha)} & R_{2112}^{(\alpha)} & R_{2121}^{(\alpha)} & R_{2122}^{(\alpha)} \\ R_{2211}^{(\alpha)} & R_{2212}^{(\alpha)} & R_{2221}^{(\alpha)} & R_{2222}^{(\alpha)} \end{pmatrix} \quad ...(3.3)$$

where

$$R_{1111}^{(\alpha)} = R_{1112}^{(\alpha)} = R_{1121}^{(\alpha)} = R_{1122}^{(\alpha)} = R_{2211}^{(\alpha)} = R_{2212}^{(\alpha)} = R_{2221}^{(\alpha)} = R_{2222}^{(\alpha)} = 0 \quad ...(3.4)$$

$$R_{1211}^{(\alpha)} = \frac{2(1-\alpha)}{\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)^2}\left[\left(\left[\left(-\frac{1}{\nu^3}+\frac{1}{8}\psi_2\left(\frac{\nu}{2}\right)+\frac{(\nu+1)}{16}\psi_3\left(\frac{\nu}{2}\right)\right)\right]\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)\right.\right.$$

$$-(\nu+1)\left(\frac{1}{\nu^2}+\frac{1}{8}\psi_2\left(\frac{\nu}{2}\right)\right)\left(\psi_1\left(\frac{\nu}{2}\right)+\frac{\nu}{2}\psi_2\left(\frac{\nu}{2}\right)\right)+\left(\frac{\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)}{4D_0^2}\right)\Bigg)(-\frac{1}{\nu}+\frac{1}{4}\psi_1(\frac{\nu}{2})-\frac{1}{4D_0})$$

$$\left.+\frac{(1-\alpha)\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)\left(\frac{1}{\nu^2}+\frac{1}{8}\psi_2\left(\frac{\nu}{2}\right)\right)}{8D_0}(\frac{1}{D_0}-1)]\right] \quad ...(3.5)$$

$$R_{1212}^{(\alpha)} = [\left(\frac{(D_0-\nu)(1-\alpha)}{(4D_0^2(\nu\psi_1\left(\frac{\nu}{2}\right)-5)^2}\right)\left(\left(-\frac{1}{2D_0}\right)\left(\psi_1\left(\frac{\nu}{2}\right)+\frac{\nu}{2}\psi_2\left(\frac{\nu}{2}\right)\right)+\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)\left(\frac{1}{\nu^2}-\frac{\nu}{2}\psi_2\left(\frac{\nu}{2}\right)\right)\right.$$

$$\left.+\left(\frac{1}{\nu}-\frac{1}{4}\psi_1\left(\frac{\nu}{2}\right)\right)\left(\psi_1\left(\frac{\nu}{2}\right)+\frac{\nu}{2}\psi_2\left(\frac{\nu}{2}\right)\right)\right)$$

$$+(\frac{(1-\alpha)^2}{64\Delta D_0^4})((\nu-\frac{2}{D_0})(\frac{1}{\nu^2}+\frac{1}{8}\psi_2\left(\frac{\nu}{2}\right))+(\frac{1}{\nu}-\frac{1}{4}\psi_1\left(\frac{\nu}{2}\right))] \quad ...(3.6)$$

Where $\psi_2(z) = \frac{d}{dz}(\psi_1)$

$$R_{1221}^{(\alpha)} = (1-\alpha)$$

$$\left[\left(-\frac{1}{2\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)D_0^2}-\frac{1}{2D_0^2}(\frac{[(-2\psi_1\left(\frac{\nu}{2}\right)-\nu\psi_2\left(\frac{\nu}{2}\right))-D_0]\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)-[(9-2\nu\psi_1\left(\frac{\nu}{2}\right)))-\nu D_0]\left(\psi_1\left(\frac{\nu}{2}\right)+\frac{\nu}{2}\psi_2\left(\frac{\nu}{2}\right)\right)}{\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)}\right)(-\frac{1}{\nu}+\frac{1}{4}\psi_1(\frac{\nu}{2})\right.$$

$$\left.-\frac{1}{4D_0})+\frac{(1-\alpha)}{4D_0^2}\left(\left[\frac{\left(-\frac{1}{\nu}+\frac{1}{4}\psi_1\left(\frac{\nu}{2}\right)\right)}{\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)}\right]+[\frac{\nu(\frac{1}{\nu^2}+\frac{1}{8}\psi_2(\frac{\nu}{2}))}{2\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)}]\right)\right] \quad ...(3.7)$$

$$R_{1222}^{(\alpha)}=\frac{(1-\alpha)}{D_0^2\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)^2}\left[\left(\frac{\left(5-\nu\psi_1\left(\frac{\nu}{2}\right)\right)}{2}+\frac{1}{2}\left(\begin{matrix}\left[\left(-\nu\psi_2\left(\frac{\nu}{2}\right)-2\psi_1\left(\frac{\nu}{2}\right)\right)D_0-1\right]\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)-\left[\left(9-2\nu\psi_1\left(\frac{\nu}{2}\right)\right)D_0-\nu\right]\\ \left(\psi_1\left(\frac{\nu}{2}\right)+\frac{\nu}{2}\psi_2\left(\frac{\nu}{2}\right)\right)\end{matrix}\right)\right)\begin{matrix}\left(\frac{1}{4D_0}-\frac{\nu}{4D_0^2}\right)\\ +(1-\alpha)\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)\\ \left(\left[\frac{1}{4}\left(-\frac{1}{\nu}+\frac{1}{4}\psi_1\left(\frac{\nu}{2}\right)\right)\right]+\left[\frac{\nu}{8D_0^2}\left(9-2\nu\psi_1\left(\frac{\nu}{2}\right)\right)\right]\right)\end{matrix}\right]$$

...(3.8)

$$R_{2111}^{(\alpha)}=\frac{2(1-\alpha)}{\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)^2}$$

$$\Bigg[\Bigg(\Big[\left(\psi_1\left(\frac{\nu}{2}\right)+\frac{\nu}{2}\psi_2\left(\frac{\nu}{2}\right)\right)+\left(\frac{1}{\nu^2}+\frac{1}{4}\psi_2\left(\frac{\nu}{2}\right)\right)D_0\Big]\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)-\Big[\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)$$

$$+\left(-\frac{1}{\nu}+\frac{1}{4}\psi_1\left(\frac{\nu}{2}\right)\right)D_0\Big]\left(\psi_1\left(\frac{\nu}{2}\right)+\frac{\nu}{2}\psi_2\left(\frac{\nu}{2}\right)\right)\Bigg)\frac{\left(-\frac{1}{\nu}+\frac{1}{4}\psi_1\left(\frac{\nu}{2}\right)-\frac{1}{4D_0}\right)}{D_0}$$

$$+\frac{(1-\alpha)\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)}{64D_0^2}\left(\left[\left(\frac{1}{\nu^2}+\frac{1}{8}\psi_2\left(\frac{\nu}{2}\right)\right)D_0-2\left(-\frac{1}{\nu}+\frac{1}{4}\psi_1\left(\frac{\nu}{2}\right)\right)\right]\right)\Bigg]$$

...(3.9)

$$R_{2112}^{(\alpha)}=\frac{(1-\alpha)}{\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)^2}$$

$$\left[\left(\left(4\left(\frac{1}{\nu^2}+\frac{1}{8}\psi_2\left(\frac{\nu}{2}\right)\right)D_0\right)\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)-\left(1+4\left(-\frac{1}{\nu}+\frac{1}{4}\psi_1\left(\frac{\nu}{2}\right)\right)D_0\right)\left(\psi_1\left(\frac{\nu}{2}\right)+\frac{\nu}{2}\psi_2\left(\frac{\nu}{2}\right)\right)\right)\left(-\frac{1}{4D_0}+\frac{\nu}{4D_0^2}\right)\right.$$

$$\left.+\left(\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)\frac{(1-\alpha)}{16D_0^3}\left(\left[4D_0\left(\frac{1}{\nu^2}+\frac{1}{8}\psi_2\left(\frac{\nu}{2}\right)\right)\right]-\left[1-8\nu\left(-\frac{1}{\nu}+\frac{1}{4}\psi_1\left(\frac{\nu}{2}\right)\right)\right]\right)\right)\right]$$

...(3.10)

$$R_{2121}^{(\alpha)}=\frac{(\alpha-1)}{\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)^2}$$

$$\left[\left(\left(\frac{\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)+\nu\left(\psi_1\left(\frac{\nu}{2}\right)+\frac{\nu}{2}\psi_2\left(\frac{\nu}{2}\right)\right)}{2D_0^2}+\frac{\left[\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)\left(2\psi_1\left(\frac{\nu}{2}\right)+\nu\psi_2\left(\frac{\nu}{2}\right)\right)+\left(9-2\nu\psi_1\left(\frac{\nu}{2}\right)\right)\left(\psi_1\left(\frac{\nu}{2}\right)+\frac{\nu}{2}\psi_2\left(\frac{\nu}{2}\right)\right)\right]}{2D_0}\left(9-2\nu\psi_1\left(\frac{\nu}{2}\right)\right)\right.\right.$$

$$\left.\left.-\left[\frac{\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)}{2D_0^2}\right]\right)\left[-\frac{1}{\nu}+\frac{1}{4}\psi_1\left(\frac{\nu}{2}\right)-\frac{1}{4D_0}\right]+\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)\frac{(1-\alpha)}{32D_0^2}\left(\nu\psi_2\left(\frac{\nu}{2}\right)+2\psi_1\left(\frac{\nu}{2}\right)\right)\right]$$

...(3.11)

$$R_{2122}^{(\alpha)}=\frac{(1-\alpha)}{2D_0^2\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)^2}$$

$$\left[\left(\left(\left(-2\nu\psi_1\left(\frac{\nu}{2}\right)-\frac{\nu}{2}\psi_2\left(\frac{\nu}{2}\right)\right)D_0-1\right)\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)-\left(9-2\nu\psi_1\left(\frac{\nu}{2}\right)\right)D_0-\nu\right)\left(\psi_1\left(\frac{\nu}{2}\right)+\frac{\nu}{2}\psi_2\left(\frac{\nu}{2}\right)\right)\right)+[\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)]\right)\left(-\frac{1}{4D_0}+\frac{\nu}{4D_0^2}\right)$$

$$+\frac{(1-\alpha)}{4}\left(\left[\frac{9-2\nu\psi_1\left(\frac{\nu}{2}\right)}{D_0}\right]-[\left(-\frac{1}{\nu}+\frac{1}{4}\psi_1\left(\frac{\nu}{2}\right)\right)]\right)\Bigg] \quad ...(3.12)$$

Proof

It is obtained that the Fisher Information Matrix of GBM Manifold $[g_{ij}]$ (Mageed, 2024) is determined by:

$$[g_{ij}]=\begin{pmatrix}-\frac{1}{\nu}+\frac{1}{4}\psi_1(\frac{\nu}{2}) & -\frac{1}{4D_0}\\ -\frac{1}{4D_0} & \frac{\nu}{4D_0^2}\end{pmatrix}, \text{with } \Delta=\det([g_{ij}])=\frac{1}{16D_0^2}\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)\neq 0 \text{ (Mageed, 2024)} \quad ...(3.13)$$

Moreover, the reader can check that the α-connections of the GBM manifold (Mageed, 2024) given by

$$\Gamma_{11,1}^{(\alpha)}=\frac{(1-\alpha)}{2}(\frac{1}{\nu^2}+\frac{1}{8}\psi_2(\frac{\nu}{2})) \quad ...(3.14)$$

$$\Gamma_{11,2}^{(\alpha)}=\Gamma_{12,1}^{(\alpha)}=\Gamma_{21,1}^{(\alpha)}=0 \quad ...(3.15)$$

$$\Gamma_{12,2}^{(\alpha)}=\frac{(1-\alpha)}{8D_0^2}=\Gamma_{22,1}^{(\alpha)}=\Gamma_{21,2}^{(\alpha)} \quad ...(3.16)$$

$$\Gamma_{22,2}^{(\alpha)}=\frac{(\alpha-1)\nu}{4D_0^3} \quad ...(3.17)$$

$$\Gamma_{11}^{1(\alpha)}=\frac{(1-\alpha)\nu}{8\Delta D_0^2}(\frac{1}{\nu^2}+\frac{1}{8}\psi_2(\frac{\nu}{2})),\Gamma_{11}^{1(0)}=\frac{\nu}{8\Delta D_0^2}(\frac{1}{\nu^2}+\frac{1}{8}\psi_2(\frac{\nu}{2})) \quad ...(3.18)$$

$$\Gamma_{12}^{1(\alpha)}=\Gamma_{21}^{1(\alpha)}=\frac{(1-\alpha)}{32\Delta D_0^3},\Gamma_{12}^{1(0)}=\Gamma_{21}^{1(0)}=\frac{1}{32\Delta D_0^3} \quad ...(3.19)$$

$$\Gamma_{22}^{1(\alpha)}=\frac{(\alpha-1)\nu}{32\Delta D_0^4},\Gamma_{22}^{1(0)}=-\frac{\nu}{32\Delta D_0^4} \quad ...(3.20)$$

$$\Gamma_{11}^{2(\alpha)}=\frac{(1-\alpha)}{8\Delta D_0^2}(\frac{1}{\nu^2}+\frac{1}{8}\psi_2(\frac{\nu}{2})),\ \Gamma_{11}^{2(0)}=\frac{1}{8\Delta D_0^2}(\frac{1}{\nu^2}+\frac{1}{8}\psi_2(\frac{\nu}{2})) \quad ...(3.21)$$

$$\Gamma_{12}^{2(\alpha)}=\Gamma_{21}^{2(\alpha)}=\frac{(1-\alpha)}{8D_0^2\Delta}\left(-\frac{1}{\nu}+\frac{1}{4}\psi_1\left(\frac{\nu}{2}\right)\right),\Gamma_{12}^{2(0)}=\Gamma_{21}^{2(0)}=\frac{1}{8\Delta D_0^2}\left(-\frac{1}{\nu}+\frac{1}{4}\psi_1\left(\frac{\nu}{2}\right)\right) \quad ...(3.22)$$

$$\Gamma_{22}^{2(\alpha)}=\frac{(1-\alpha)}{32\Delta D_0^3}(9-2\nu\psi_1\left(\frac{\nu}{2}\right)),\ \Gamma_{22}^{2(0)}=\frac{1}{32\Delta D_0^3}(9-2\nu\psi_1\left(\frac{\nu}{2}\right)) \quad ...(3.23)$$

It can be verified that:

$$R_{1111}^{(\alpha)}=R_{2222}^{(\alpha)}=R_{1112}^{(\alpha)}=R_{1121}^{(\alpha)}=R_{1122}^{(\alpha)}=R_{2211}^{(\alpha)}=R_{2212}^{(\alpha)}=R_{2221}^{(\alpha)}=0 \text{ (c.f., (3.4))}$$

$$R_{1211}^{(\alpha)}=\left[\left(\frac{\partial}{\partial\nu}(\Gamma_{11}^{1(\alpha)}+\Gamma_{11}^{2(\alpha)})-\frac{\partial}{\partial D_0}(\Gamma_{21}^{1(\alpha)}+\Gamma_{21}^{2(\alpha)})\right)(g_{11}+g_{21})+\left([\Gamma_{21,1}^{(\alpha)}\Gamma_{11}^{1(\alpha)}+\Gamma_{22,1}^{(\alpha)}\Gamma_{11}^{2(\alpha)}]-[\Gamma_{11,1}^{(\alpha)}\Gamma_{21}^{1(\alpha)}+\Gamma_{12,1}^{(\alpha)}\Gamma_{21}^{2(\alpha)}]\right)\right]$$

$$=\left[\left(\frac{\partial}{\partial\nu}(\frac{2(1-\alpha)\nu}{\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)}(\frac{1}{\nu^2}+\frac{1}{8}\psi_2(\frac{\nu}{2}))+\frac{2(1-\alpha)}{\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)}(\frac{1}{\nu^2}+\frac{1}{8}\psi_2(\frac{\nu}{2})))-\frac{\partial}{\partial D_0}(\frac{(1-\alpha)}{32\Delta D_0^3}+\frac{2(1-\alpha)}{\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)}\left(-\frac{1}{\nu}+\frac{1}{4}\psi_1\left(\frac{\nu}{2}\right)\right))\right)(-\frac{1}{\nu}+\frac{1}{4}\psi_1(\frac{\nu}{2}$$

$$-\frac{1}{4D_0})+\left(\left[\frac{(1-\alpha)}{4D_0^2}\frac{(1-\alpha)}{\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)}\left(\frac{1}{\nu^2}+\frac{1}{8}\psi_2\left(\frac{\nu}{2}\right)\right)\right]-[\frac{(1-\alpha)}{2}(\frac{1}{\nu^2}+\frac{1}{8}\psi_2(\frac{\nu}{2}))\frac{(1-\alpha)}{2D_0\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)}]\right)\right]$$

$$= \frac{2(1-\alpha)}{\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)^2}$$

$$\left[\left(\left[\left(-\frac{1}{\nu^3}+\frac{1}{8}\psi_2\left(\frac{\nu}{2}\right)+\frac{(\nu+1)}{16}\psi_3\left(\frac{\nu}{2}\right)\right)\right]\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)-(\nu+1)\left(\frac{1}{\nu^2}+\frac{1}{8}\psi_2\left(\frac{\nu}{2}\right)\right)\left(\psi_1\left(\frac{\nu}{2}\right)+\frac{\nu}{2}\psi_2\left(\frac{\nu}{2}\right)\right)+\left(\frac{\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)}{4D_0^2}\right)\right)(-\frac{1}{\nu}+\frac{1}{4}\psi_1(\frac{\nu}{2}) \right.$$

$$\left. -\frac{1}{4D_0})+\frac{(1-\alpha)\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)\left(\frac{1}{\nu^2}+\frac{1}{8}\psi_2\left(\frac{\nu}{2}\right)\right)}{8D_0}(\frac{1}{D_0}-1)]\right] \text{(c.f., (3.5))}$$

$$R_{1212}^{(\alpha)} = [\left(\frac{(D_0-\nu)(1-\alpha)}{(4D_0^2(\nu\psi_1\left(\frac{\nu}{2}\right)-5)^2}\right)\left(\left(-\frac{1}{2D_0}\right)\left(\psi_1\left(\frac{\nu}{2}\right)+\frac{\nu}{2}\psi_2\left(\frac{\nu}{2}\right)\right)+\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)\left(\frac{1}{\nu^2}-\frac{\nu}{2}\psi_2\left(\frac{\nu}{2}\right)\right)+\left(\frac{1}{\nu}-\frac{1}{4}\psi_1\left(\frac{\nu}{2}\right)\right)\left(\psi_1\left(\frac{\nu}{2}\right)+\frac{\nu}{2}\psi_2\left(\frac{\nu}{2}\right)\right)\right)$$

$$(\nu-\frac{2}{D_0})(\frac{1}{\nu^2}+\frac{1}{8}\psi_2\left(\frac{\nu}{2}\right))+(\frac{1}{\nu}-\frac{1}{4}\psi_1\left(\frac{\nu}{2}\right)))] \text{(c.f., (3.6))}$$

$$R_{1221}^{(\alpha)} = [\left(\partial_2(\Gamma_{12}^{1(\alpha)}+\Gamma_{12}^{2(\alpha)})-\partial_1(\Gamma_{22}^{1(\alpha)}+\Gamma_{22}^{2(\alpha)})\right)(g_{11}+g_{21})+\left([\Gamma_{21,1}^{(\alpha)}\Gamma_{12}^{1(\alpha)}+\Gamma_{22,1}^{(\alpha)}\Gamma_{12}^{2(\alpha)}]-[\Gamma_{11,1}^{(\alpha)}\Gamma_{22}^{1(\alpha)}+\Gamma_{12,1}^{(\alpha)}\Gamma_{22}^{2(\alpha)}]\right)]$$

$$=(1-\alpha)\left[\left(-\frac{1}{2\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)D_0^2}-\frac{1}{2D_0^2}(\frac{[(-2\psi_1\left(\frac{\nu}{2}\right)-\nu\psi_2\left(\frac{\nu}{2}\right))-D_0]\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)-[(9-2\nu\psi_1\left(\frac{\nu}{2}\right)))-\nu D_0]\left(\psi_1\left(\frac{\nu}{2}\right)+\frac{\nu}{2}\psi_2\left(\frac{\nu}{2}\right)\right)}{\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)}\right)\right.$$

$$\left.(-\frac{1}{\nu}+\frac{1}{4}\psi_1(\frac{\nu}{2})-\frac{1}{4D_0})+\frac{(1-\alpha)}{4D_0^2}\left(\left[\frac{\left(-\frac{1}{\nu}+\frac{1}{4}\psi_1\left(\frac{\nu}{2}\right)\right)}{\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)}\right]+[\frac{\nu(\frac{1}{\nu^2}+\frac{1}{8}\psi_2(\frac{\nu}{2}))}{2\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)}]\right)\right] \text{(c.f., (3.7))}$$

Engaging the same procedure, the remaining tensorial components will follow.

Theorem 3.3: The α-Sectional Curvature Tensor, ב of the GBM Manifold reads

$$ב = \begin{pmatrix} K_{1111}^{(\alpha)} & K_{1112}^{(\alpha)} & K_{1121}^{(\alpha)} & K_{1122}^{(\alpha)} \\ K_{1211}^{(\alpha)} & K_{1212}^{(\alpha)} & K_{1221}^{(\alpha)} & K_{1222}^{(\alpha)} \\ K_{2111}^{(\alpha)} & K_{2112}^{(\alpha)} & K_{2121}^{(\alpha)} & K_{2122}^{(\alpha)} \\ K_{2211}^{(\alpha)} & K_{2212}^{(\alpha)} & K_{2221}^{(\alpha)} & K_{2222}^{(\alpha)} \end{pmatrix} \quad ...(3.23)$$

where

$$K_{1111}^{(\alpha)} = K_{1112}^{(\alpha)} = K_{1121}^{(\alpha)} = K_{1122}^{(\alpha)} = K_{2211}^{(\alpha)} = K_{2212}^{(\alpha)} = K_{2221}^{(\alpha)} = K_{2222}^{(\alpha)} = 0 \quad ...(3.24)$$

$$K_{1211}^{(\alpha)} = \frac{32(1-\alpha)D_0^2}{\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)^3}\left[\left(\left[\left(-\frac{1}{\nu^3}+\frac{1}{8}\psi_2\left(\frac{\nu}{2}\right)+\frac{(\nu+1)}{16}\psi_3\left(\frac{\nu}{2}\right)\right)\right]\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)-(\nu+1)\left(\frac{1}{\nu^2}+\frac{1}{8}\psi_2\left(\frac{\nu}{2}\right)\right)\left(\psi_1\left(\frac{\nu}{2}\right)+\frac{\nu}{2}\psi_2\left(\frac{\nu}{2}\right)\right)\right.\right.$$

$$\left.\left.+\left(\frac{\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)}{4D_0^2}\right)\right)(-\frac{1}{\nu}+\frac{1}{4}\psi_1(\frac{\nu}{2})-\frac{1}{4D_0})+\frac{(1-\alpha)\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)\left(\frac{1}{\nu^2}+\frac{1}{8}\psi_2\left(\frac{\nu}{2}\right)\right)}{8D_0}(\frac{1}{D_0}-1)]\right] \quad ...(3.25)$$

$$K_{1212}^{(\alpha)}=[\left(\frac{4(D_0-\nu)(1-\alpha)}{(\nu\psi_1\left(\frac{\nu}{2}\right)-5)^3}\right)\left(\left(-\frac{1}{2D_0}\right)\left(\psi_1\left(\frac{\nu}{2}\right)+\frac{\nu}{2}\psi_2\left(\frac{\nu}{2}\right)\right)+\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)\left(\frac{1}{\nu^2}-\frac{\nu}{2}\psi_2\left(\frac{\nu}{2}\right)\right)+\left(\frac{1}{\nu}-\frac{1}{4}\psi_1\left(\frac{\nu}{2}\right)\right)\left(\psi_1\left(\frac{\nu}{2}\right)+\frac{\nu}{2}\psi_2\left(\frac{\nu}{2}\right)\right)\right)+$$

$$(\nu-\frac{2}{D_0})(\frac{1}{\nu^2}+\frac{1}{8}\psi_2\left(\frac{\nu}{2}\right))+(\frac{1}{\nu}-\frac{1}{4}\psi_1\left(\frac{\nu}{2}\right))]] \quad ...(3.26)$$

$$K_{1221}^{(\alpha)}=\frac{16D_0^2(1-\alpha)}{(\nu\psi_1\left(\frac{\nu}{2}\right)-5)^3}$$

$$\left[\left(-\frac{1}{2\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)D_0^2}-\frac{1}{2D_0^2}\left(\frac{[(-2\psi_1\left(\frac{\nu}{2}\right)-\nu\psi_2\left(\frac{\nu}{2}\right))-D_0]\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)-[(9-2\nu\psi_1\left(\frac{\nu}{2}\right)))-\nu D_0]\left(\psi_1\left(\frac{\nu}{2}\right)+\frac{\nu}{2}\psi_2\left(\frac{\nu}{2}\right)\right)}{\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)}\right)(-\frac{1}{\nu}+\frac{1}{4}\psi_1(\frac{\nu}{2})\right.$$

$$\left.-\frac{1}{4D_0})+\frac{(1-\alpha)}{4D_0^2}\left(\left[\frac{\left(-\frac{1}{\nu}+\frac{1}{4}\psi_1\left(\frac{\nu}{2}\right)\right)}{\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)}\right]+[\frac{\nu(\frac{1}{\nu^2}+\frac{1}{8}\psi_2(\frac{\nu}{2}))}{2\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)}]\right)\right] \quad ...(3.27)$$

$$K_{1222}^{(\alpha)}=\frac{16(1-\alpha)}{\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)^3}\left[\begin{array}{c}\left(\frac{\left(5-\nu\psi_1\left(\frac{\nu}{2}\right)\right)}{2}+\frac{1}{2}\left(\begin{array}{c}\left[\left(-\nu\psi_2\left(\frac{\nu}{2}\right)-2\psi_1\left(\frac{\nu}{2}\right)\right)D_0-1\right]\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)-\left[\left(9-2\nu\psi_1\left(\frac{\nu}{2}\right)\right)D_0-\nu\right]\\\left(\psi_1\left(\frac{\nu}{2}\right)+\frac{\nu}{2}\psi_2\left(\frac{\nu}{2}\right)\right)\end{array}\right)\right)\\\left(\frac{1}{4D_0}-\frac{\nu}{4D_0^2}\right)\\+(1-\alpha)\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)\\\left(\left[\frac{1}{4}\left(-\frac{1}{\nu}+\frac{1}{4}\psi_1\left(\frac{\nu}{2}\right)\right)\right]+[\frac{\nu}{8D_0^2}(9-2\nu\psi_1\left(\frac{\nu}{2}\right))]\right)\end{array}\right]$$

...(3.28)

$$K_{2111}^{(\alpha)}=\frac{32D_0^2(1-\alpha)}{\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)^3}\left[\left(\left[\left(\psi_1\left(\frac{\nu}{2}\right)+\frac{\nu}{2}\psi_2\left(\frac{\nu}{2}\right)\right)+\left(\frac{1}{\nu^2}+\frac{1}{4}\psi_2\left(\frac{\nu}{2}\right)\right)D_0\right]\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)-[\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)\right.\right.$$

$$+\left(-\frac{1}{\nu}+\frac{1}{4}\psi_1\left(\frac{\nu}{2}\right)\right)D_0]\left(\psi_1\left(\frac{\nu}{2}\right)+\frac{\nu}{2}\psi_2\left(\frac{\nu}{2}\right)\right)\Bigg)\frac{(-\frac{1}{\nu}+\frac{1}{4}\psi_1\left(\frac{\nu}{2}\right)-\frac{1}{4D_0})}{D_0}$$

$$\left.+\frac{(1-\alpha)\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)}{64D_0^2}\left(\left[\left(\frac{1}{\nu^2}+\frac{1}{8}\psi_2\left(\frac{\nu}{2}\right)\right)D_0-2\left(-\frac{1}{\nu}+\frac{1}{4}\psi_1\left(\frac{\nu}{2}\right)\right)\right]\right)\right] \quad ...(3.29)$$

$$K_{2112}^{(\alpha)}=\frac{16D_0^2(1-\alpha)}{\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)^3}\left[\left(\left(4\left(\frac{1}{\nu^2}+\frac{1}{8}\psi_2\left(\frac{\nu}{2}\right)\right)D_0\right)\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)-\left(1+4\left(-\frac{1}{\nu}+\frac{1}{4}\psi_1\left(\frac{\nu}{2}\right)\right)D_0\right)\left(\psi_1\left(\frac{\nu}{2}\right)+\frac{\nu}{2}\psi_2\left(\frac{\nu}{2}\right)\right)\right)\left(-\frac{1}{4D_0}+\frac{\nu}{4D_0^2}\right)\right.$$

$$\left.+\left(\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)\frac{(1-\alpha)}{16D_0^3}(\left[4D_0\left(\frac{1}{\nu^2}+\frac{1}{8}\psi_2\left(\frac{\nu}{2}\right)\right)\right]-[1-8\nu\left(-\frac{1}{\nu}+\frac{1}{4}\psi_1\left(\frac{\nu}{2}\right)\right)]\right)\right] \quad ...(3.30)$$

$$K_{2121}^{(\alpha)}=\frac{16D_0^2(\alpha-1)}{\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)^3}$$

$$\Bigg[\Bigg(\Bigg(\Bigg(\frac{\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)+\nu(\psi_1\left(\frac{\nu}{2}\right)+\frac{\nu}{2}\psi_2\left(\frac{\nu}{2}\right))}{2D_0^2}+\frac{[\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)\left(2\psi_1\left(\frac{\nu}{2}\right)+\nu\psi_2\left(\frac{\nu}{2}\right)\right)+(9-2\nu\psi_1\left(\frac{\nu}{2}\right)\left(\psi_1\left(\frac{\nu}{2}\right)+\frac{\nu}{2}\psi_2\left(\frac{\nu}{2}\right)\right)]}{2D_0}\left(9-2\nu\psi_1\left(\frac{\nu}{2}\right)\right)$$

$$-[\frac{\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)}{2D_0^2}]\Bigg)\left[-\frac{1}{\nu}+\frac{1}{4}\psi_1\left(\frac{\nu}{2}\right)-\frac{1}{4D_0}\right]+\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)\frac{(1-\alpha)}{32D_0^2}\left(\nu\psi_2\left(\frac{\nu}{2}\right)+2\psi_1\left(\frac{\nu}{2}\right)\right)\Bigg] \quad ...(3.31)$$

$$K_{2122}^{(\alpha)}=\frac{8(1-\alpha)}{\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)^3}\Bigg[\Bigg(\Bigg(\left(\left(-2\nu\psi_1\left(\frac{\nu}{2}\right)-\frac{\nu}{2}\psi_2\left(\frac{\nu}{2}\right)\right)D_0-1\right)\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)-\left(9-2\nu\psi_1\left(\frac{\nu}{2}\right)\right)D_0-\nu)\left(\psi_1\left(\frac{\nu}{2}\right)+\frac{\nu}{2}\psi_2\left(\frac{\nu}{2}\right)\right)\Bigg)$$

$$+[\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)]\Bigg)\left(-\frac{1}{4D_0}+\frac{\nu}{4D_0^2}\right)+\frac{(1-\alpha)}{4}\left(\left[\frac{9-2\nu\psi_1\left(\frac{\nu}{2}\right)}{D_0}\right]-[\left(-\frac{1}{\nu}+\frac{1}{4}\psi_1\left(\frac{\nu}{2}\right)\right)]\right)\Bigg] \quad ...(3.32)$$

Proof

Engaging

$$K_{ijij}^{(\alpha)}=\frac{R_{ijij}^{(\alpha)}}{\Delta}, i,j=1,2 \text{ (c.f., (2.3))}$$

and (3.4)-(3.12), the proofs are immediate.

4. Ricci Curvature Tensor, Υ, Curvature of Spacetime (Einestein Tensor), $\wp$ and Stress Energy Tensor, Ω of GBM Manifold

4.1. Ricci, Einstein, and Stress Energy Tensors

Theorem 4.1: The Ricci Tensor corresponding to the curvature parameter $\alpha = 0$, Υ of the GBM manifold is determined by:

$$\Upsilon=\begin{pmatrix}R_{11}^{(0)} & R_{12}^{(0)}\\ R_{21}^{(0)} & R_{22}^{(0)}\end{pmatrix} \quad ...(4.1)$$

where

$$R_{11}^{(0)}=\frac{4}{D_0(\nu\psi_1\left(\frac{\nu}{2}\right)-5)}[\left(\frac{1}{4}\psi_1\left(\frac{\nu}{2}\right)-\frac{1}{4D_0}-\frac{1}{\nu}\right)\Bigg(\frac{\left(\psi_1\left(\frac{\nu}{2}\right)+\frac{\nu}{2}\psi_2\left(\frac{\nu}{2}\right)\right)}{D_0\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)^2}+\frac{\left(\frac{1}{\nu^2}+\frac{1}{8}\psi_2\left(\frac{\nu}{2}\right)\right)\left(\psi_2\left(\frac{\nu}{2}\right)+\frac{8}{\nu^2}-1\right)}{D_0^2\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)}$$

$$+\frac{4\nu}{\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)}\left(\frac{(D_0-\nu)}{4D_0^2\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)^2}\left(-\frac{1}{2D_0}\right)\left(\psi_1\left(\frac{\nu}{2}\right)+\frac{\nu}{2}\psi_2\left(\frac{\nu}{2}\right)\right)+\left(\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)\right)\left(\frac{1}{\nu^2}-\frac{\nu}{2}\psi_2\left(\frac{\nu}{2}\right)\right)\right.$$

$$\left.+\left(\frac{1}{\nu}-\frac{1}{4}\psi_1\left(\frac{\nu}{2}\right)\right)\left(\psi_1\left(\frac{\nu}{2}\right)+\frac{\nu}{2}\psi_2\left(\frac{\nu}{2}\right)\right)\right)+\frac{4(\left(\nu-\frac{2}{D_0}\right)(\frac{1}{\nu^2}+\frac{1}{8}\psi_2\left(\frac{\nu}{2}\right)+\frac{1}{\nu}-\frac{1}{4}\psi_1\left(\frac{\nu}{2}\right))}{D_0^2\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)}\Bigg] \text{ (Mageed, 2024)} \quad ...(4.2)$$

$R_{12}^{(0)}$ = We have

$$R_{ik}^{(\alpha)}=R_{ijkl}^{(\alpha)}g^{jl}, i,j,k,l=1,2,3,\ldots,n \text{ (c.f., (2.2))}$$

$$R_{12}^{(0)} = \Bigg[\Bigg(-\frac{1}{2\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)D_0^2} - \frac{1}{2D_0^2}\Bigg(\frac{[(-2\psi_1\left(\frac{\nu}{2}\right)-\nu\psi_2\left(\frac{\nu}{2}\right))-D_0]\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)-[(9-2\nu\psi_1\left(\frac{\nu}{2}\right)))-\nu D_0]\left(\psi_1\left(\frac{\nu}{2}\right)+\frac{\nu}{2}\psi_2\left(\frac{\nu}{2}\right)\right)}{\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)}\Bigg)\Big(-\frac{1}{\nu}+\frac{1}{4}\psi_1\left(\frac{\nu}{2}\right)$$

$$-\frac{1}{4D_0}\Big)+\frac{1}{4D_0^2}\left(\left[\frac{\left(-\frac{1}{\nu}+\frac{1}{4}\psi_1\left(\frac{\nu}{2}\right)\right)}{\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)}\right]+\left[\frac{\nu\left(\frac{1}{\nu^2}+\frac{1}{8}\psi_2\left(\frac{\nu}{2}\right)\right)}{2\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)}\right]\right)\Bigg]$$

$$\frac{4D_0}{\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)} + \Bigg(\frac{16\left(-\frac{1}{\nu}+\frac{1}{4}\psi_1\left(\frac{\nu}{2}\right)\right)}{\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)^3}\Bigg[\Bigg(\frac{\left(5-\nu\psi_1\left(\frac{\nu}{2}\right)\right)}{2}+\frac{1}{2}\Bigg(\left[\left(-\nu\psi_2\left(\frac{\nu}{2}\right)-2\psi_1\left(\frac{\nu}{2}\right)\right)D_0-1\right]\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)-\left[\left(9-2\nu\psi_1\left(\frac{\nu}{2}\right)\right)D_0-\nu\right]\left(\psi_1\left(\frac{\nu}{2}\right)+\frac{\nu}{2}\psi_2\left(\frac{\nu}{2}\right)\right)\Bigg)\Bigg)\left(\frac{1}{4D_0}-\frac{\nu}{4D_0^2}\right)+\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)\left(\left[\frac{1}{4}\left(-\frac{1}{\nu}+\frac{1}{4}\psi_1\left(\frac{\nu}{2}\right)\right)\right]+\left[\frac{\nu}{8D_0^2}\left(9-2\nu\psi_1\left(\frac{\nu}{2}\right)\right)\right]\right)\Bigg]\Bigg)$$

...(4.3)

$$R_{21}^{(0)} = \frac{8\nu}{\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)^3}$$

$$\Bigg[\Bigg(\Bigg[\left(\psi_1\left(\frac{\nu}{2}\right)+\frac{\nu}{2}\psi_2\left(\frac{\nu}{2}\right)\right)+\left(\frac{1}{\nu^2}+\frac{1}{4}\psi_2\left(\frac{\nu}{2}\right)\right)D_0\Bigg]\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)-[\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)$$

$$+\left(-\frac{1}{\nu}+\frac{1}{4}\psi_1\left(\frac{\nu}{2}\right)\right)D_0]\left(\psi_1\left(\frac{\nu}{2}\right)+\frac{\nu}{2}\psi_2\left(\frac{\nu}{2}\right)\right)\Bigg)\frac{\left(-\frac{1}{\nu}+\frac{1}{4}\psi_1\left(\frac{\nu}{2}\right)-\frac{1}{4D_0}\right)}{D_0}$$

$$+\frac{\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)}{64D_0^2}\left(\left[\left(\frac{1}{\nu^2}+\frac{1}{8}\psi_2\left(\frac{\nu}{2}\right)\right)D_0-2\left(-\frac{1}{\nu}+\frac{1}{4}\psi_1\left(\frac{\nu}{2}\right)\right)\right]\right)\Bigg] \quad ...(4.4)$$

$$R_{22}^{(0)} = \frac{2}{\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)^3}\Bigg[\frac{1}{D_0}\Bigg[\Bigg(\Bigg(\left(\left(-2\nu\psi_1\left(\frac{\nu}{2}\right)-\frac{\nu}{2}\psi_2\left(\frac{\nu}{2}\right)\right)D_0-1\right)\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)-\left(9-2\nu\psi_1\left(\frac{\nu}{2}\right)\right)D_0-\nu\Big)\left(\psi_1\left(\frac{\nu}{2}\right)+\frac{\nu}{2}\psi_2\left(\frac{\nu}{2}\right)\right)\Bigg)$$

$$+\left[\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)\right]\Bigg)\left(-\frac{1}{4D_0}+\frac{\nu}{4D_0^2}\right)+\frac{1}{4}\left(\left[\frac{9-2\nu\psi_1\left(\frac{\nu}{2}\right)}{D_0}\right]-\left[\left(-\frac{1}{\nu}+\frac{1}{4}\psi_1\left(\frac{\nu}{2}\right)\right)\right]\right)\Bigg]-2\nu$$

$$\Bigg[\Bigg(\Bigg(\frac{\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)+\nu\left(\psi_1\left(\frac{\nu}{2}\right)+\frac{\nu}{2}\psi_2\left(\frac{\nu}{2}\right)\right)}{2D_0^2}+\frac{[\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)\left(2\psi_1\left(\frac{\nu}{2}\right)+\nu\psi_2\left(\frac{\nu}{2}\right)\right)+(9-2\nu\psi_1\left(\frac{\nu}{2}\right)\left(\psi_1\left(\frac{\nu}{2}\right)+\frac{\nu}{2}\psi_2\left(\frac{\nu}{2}\right)\right)]}{2D_0}\left(9-2\nu\psi_1\left(\frac{\nu}{2}\right)\right)$$

$$-\left[\frac{\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)}{2D_0^2}\right]\Bigg)\Bigg)\left[-\frac{1}{\nu}+\frac{1}{4}\psi_1\left(\frac{\nu}{2}\right)-\frac{1}{4D_0}\right]+\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)\frac{(1)}{32D_0^2}\left(\nu\psi_2\left(\frac{\nu}{2}\right)+2\psi_1\left(\frac{\nu}{2}\right)\right)\Bigg] \quad ...(4.5)$$

Proof

It is proven (Mageed, 2024) that the Inverse of Fisher Information Matrix, IFIM of GBM manifold is given by:

$$[g_{ij}] = \frac{1}{\Delta}\begin{pmatrix} \frac{\nu}{4D_0^2} & \frac{1}{4D_0} \\ \frac{1}{4D_0} & -\frac{1}{\nu}+\frac{1}{4}\psi_1(\frac{\nu}{2}) \end{pmatrix}, \text{with } \Delta = \frac{1}{16D_0^2}\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right) \quad ...(4.6)$$

We have by (2.2),

$$R_{12}^{(0)} = R_{1121}^{(0)} g^{11} + R_{1122}^{(0)} g^{12} + R_{1221}^{(0)} g^{21} + R_{1222}^{(0)} g^{22} \quad ...(4.7)$$

$$= \Bigg[\Bigg(-\frac{1}{2\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)D_0^2} - \frac{1}{2D_0^2}\Big(\frac{[(-2\psi_1\left(\frac{\nu}{2}\right)-\nu\psi_2\left(\frac{\nu}{2}\right))-D_0]\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)-[(9-2\nu\psi_1\left(\frac{\nu}{2}\right)))-\nu D_0]\left(\psi_1\left(\frac{\nu}{2}\right)+\frac{\nu}{2}\psi_2\left(\frac{\nu}{2}\right)\right)}{\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)}\Bigg)(-\frac{1}{\nu}+\frac{1}{4}\psi_1(\frac{\nu}{2})$$

$$-\frac{1}{4D_0}) + \frac{1}{4D_0^2}\left(\left[\frac{\left(-\frac{1}{\nu}+\frac{1}{4}\psi_1\left(\frac{\nu}{2}\right)\right)}{\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)}\right] + [\frac{\nu(\frac{1}{\nu^2}+\frac{1}{8}\psi_2(\frac{\nu}{2}))}{2\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)}]\right)\Bigg]\frac{4D_0}{\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)}$$

$$+\left(\frac{16(-\frac{1}{\nu}+\frac{1}{4}\psi_1(\frac{\nu}{2})}{\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)^3}\left[\begin{matrix}\left(\frac{\left(5-\nu\psi_1\left(\frac{\nu}{2}\right)\right)}{2}+\frac{1}{2}\left(\begin{matrix}\left[\left(-\nu\psi_2\left(\frac{\nu}{2}\right)-2\psi_1\left(\frac{\nu}{2}\right)\right)D_0-1\right]\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)-\left[\left(9-2\nu\psi_1\left(\frac{\nu}{2}\right)\right)D_0-\nu\right] \\ \left(\psi_1\left(\frac{\nu}{2}\right)+\frac{\nu}{2}\psi_2\left(\frac{\nu}{2}\right)\right)\end{matrix}\right)\right) \\ \left(\frac{1}{4D_0}-\frac{\nu}{4D_0^2}\right) \\ +\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right) \\ \left(\left[\frac{1}{4}\left(-\frac{1}{\nu}+\frac{1}{4}\psi_1\left(\frac{\nu}{2}\right)\right)\right]+\left[\frac{\nu}{8D_0^2}\left(9-2\nu\psi_1\left(\frac{\nu}{2}\right)\right)\right]\right)\end{matrix}\right]\right) \text{(c.f., (4.3))}$$

Moreover,

$$R_{21}^{(0)} = R_{2111}^{(0)} g^{11} + R_{2112}^{(0)} g^{12} + R_{2211}^{(0)} g^{21} + R_{2212}^{(0)} g^{22} \quad ...(4.8)$$

$$R_{2111}^{(0)} g^{11} = \frac{8\nu}{\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)^3}$$

$$\Bigg[\Bigg(\left[\left(\psi_1\left(\frac{\nu}{2}\right)+\frac{\nu}{2}\psi_2\left(\frac{\nu}{2}\right)\right)+\left(\frac{1}{\nu^2}+\frac{1}{4}\psi_2\left(\frac{\nu}{2}\right)\right)D_0\right]\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)-[\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)$$

$$+\left(-\frac{1}{\nu}+\frac{1}{4}\psi_1\left(\frac{\nu}{2}\right)\right)D_0]\left(\psi_1\left(\frac{\nu}{2}\right)+\frac{\nu}{2}\psi_2\left(\frac{\nu}{2}\right)\right)\Bigg)\frac{(-\frac{1}{\nu}+\frac{1}{4}\psi_1\left(\frac{\nu}{2}\right)-\frac{1}{4D_0})}{D_0}$$

$$+\frac{\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)}{64D_0^2}\left(\left[\left(\frac{1}{\nu^2}+\frac{1}{8}\psi_2\left(\frac{\nu}{2}\right)\right)D_0-2\left(-\frac{1}{\nu}+\frac{1}{4}\psi_1\left(\frac{\nu}{2}\right)\right)\right]\right)\Bigg] \text{(c.f., (4.4))}$$

$$R_{22}^{(0)} = R_{2121}^{(0)} g^{11} + R_{2122}^{(0)} g^{12} + R_{2221}^{(0)} g^{21} + R_{2222}^{(0)} g^{22} \quad ...(4.9)$$

$$= R_{2121}^{(0)} g^{11} + R_{2122}^{(0)} g^{12}$$

$$= \frac{2}{\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)^3}[\frac{1}{D_0}\Bigg[\Bigg(\Bigg(\left(\left(-2\nu\psi_1\left(\frac{\nu}{2}\right)-\frac{\nu}{2}\psi_2\left(\frac{\nu}{2}\right)\right)D_0-1\right)\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)-\left(9-2\nu\psi_1\left(\frac{\nu}{2}\right)\right)D_0-\nu)\left(\psi_1\left(\frac{\nu}{2}\right)+\frac{\nu}{2}\psi_2\left(\frac{\nu}{2}\right)\right)\Bigg)$$

$$+[\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)]\Bigg)\left(-\frac{1}{4D_0}+\frac{\nu}{4D_0^2}\right)+\frac{1}{4}\left(\left[\frac{9-2\nu\psi_1\left(\frac{\nu}{2}\right)}{D_0}\right]-[\left(-\frac{1}{\nu}+\frac{1}{4}\psi_1\left(\frac{\nu}{2}\right)\right)]\right)\Bigg]-2\nu$$

$$\Bigg[\Bigg(\Bigg(\frac{\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)+\nu\left(\psi_1\left(\frac{\nu}{2}\right)+\frac{\nu}{2}\psi_2\left(\frac{\nu}{2}\right)\right)}{2D_0^2}+\frac{[\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)\left(2\psi_1\left(\frac{\nu}{2}\right)+\nu\psi_2\left(\frac{\nu}{2}\right)\right)+(9-2\nu\psi_1\left(\frac{\nu}{2}\right)\left(\psi_1\left(\frac{\nu}{2}\right)+\frac{\nu}{2}\psi_2\left(\frac{\nu}{2}\right)\right)]}{2D_0}\left(9-2\nu\psi_1\left(\frac{\nu}{2}\right)\right)\Bigg)$$

$$-[\frac{\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)}{2D_0^2}]\Bigg)\left[-\frac{1}{\nu}+\frac{1}{4}\psi_1\left(\frac{\nu}{2}\right)-\frac{1}{4D_0}\right]+\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)\frac{(1)}{32D_0^2}\left(\nu\psi_2\left(\frac{\nu}{2}\right)+2\psi_1\left(\frac{\nu}{2}\right)\right)\Bigg] \text{(c.f., (4.5))}$$

Theorem 4.2: The Curvature of **Spacetime** (Einstein tensor), $\wp$ and the Stress Energy Tensor, Ω of GBM manifold corresponding to the curvature parameter, $\alpha = 0$ are determined by:

$$\wp_{\alpha=0}=\begin{pmatrix}G_{11}^{(0)} & G_{12}^{(0)}\\ G_{21}^{(0)} & G_{22}^{(0)}\end{pmatrix},\Omega_{\alpha=0}=\begin{pmatrix}\varpi_{11}^{(0)} & \varpi_{21}^{(0)}\\ \varpi_{12}^{(0)} & \varpi_{22}^{(0)}\end{pmatrix} \quad ...(4.10)$$

where

$$G_{11}^{(0)}=\frac{4}{D_0\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)}\;[\left(\frac{1}{4}\psi_1\left(\frac{\nu}{2}\right)-\frac{1}{4D_0}-\frac{1}{\nu}\right)$$

$$\left(\frac{\left(\psi_1\left(\frac{\nu}{2}\right)+\frac{\nu}{2}\psi_2\left(\frac{\nu}{2}\right)\right)}{D_0\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)^2}+\frac{\left(\frac{1}{\nu^2}+\frac{1}{8}\psi_2\left(\frac{\nu}{2}\right)\right)\left(\psi_2\left(\frac{\nu}{2}\right)+\frac{8}{\nu^2}-1\right)}{D_0^2\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)}\right.$$

$$+\frac{4\nu}{\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)}\left(\frac{(D_0-\nu)}{4D_0^2\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)^2}\left(-\frac{1}{2D_0}\right)\left(\psi_1\left(\frac{\nu}{2}\right)+\frac{\nu}{2}\psi_2\left(\frac{\nu}{2}\right)\right)+\left(\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)\right)\left(\frac{1}{\nu^2}-\frac{\nu}{2}\psi_2\left(\frac{\nu}{2}\right)\right)\right.$$

$$\left.+\left(\frac{1}{\nu}-\frac{1}{4}\psi_1\left(\frac{\nu}{2}\right)\right)\left(\psi_1\left(\frac{\nu}{2}\right)+\frac{\nu}{2}\psi_2\left(\frac{\nu}{2}\right)\right)\right)+\frac{4\left(\left(\nu-\frac{2}{D_0}\right)\left(\frac{1}{\nu^2}+\frac{1}{8}\psi_2\left(\frac{\nu}{2}\right)+\frac{1}{\nu}-\frac{1}{4}\psi_1\left(\frac{\nu}{2}\right)\right)\right)}{D_0^2\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)}\Bigg]$$

$$-\frac{2\left(\nu\psi_1\left(\frac{\nu}{2}\right)-4\right)}{\nu^2\left((ln2-3)+\psi\left(\frac{\nu}{2}\right)\right)}\left(-\frac{1}{\nu}+\frac{1}{4}\psi_1\left(\frac{\nu}{2}\right)\right)=\frac{8\pi g\,\varpi_{11}^{(0)}}{c_{light}^4} \quad ...(4.11)$$

$$G_{12}^{(0)}=\Bigg[\Bigg(-\frac{1}{2\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)D_0^2}-\frac{1}{2D_0^2}\left(\frac{[(-2\psi_1\left(\frac{\nu}{2}\right)-\nu\psi_2\left(\frac{\nu}{2}\right))-D_0]\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)-[(9-2\nu\psi_1\left(\frac{\nu}{2}\right)))-\nu D_0]\left(\psi_1\left(\frac{\nu}{2}\right)+\frac{\nu}{2}\psi_2\left(\frac{\nu}{2}\right)\right)}{\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)}\right)\Bigg)$$

$$\left(-\frac{1}{\nu}+\frac{1}{4}\psi_1\left(\frac{\nu}{2}\right)-\frac{1}{4D_0}\right)+\frac{1}{4D_0^2}\left(\left[\frac{\left(-\frac{1}{\nu}+\frac{1}{4}\psi_1\left(\frac{\nu}{2}\right)\right)}{\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)}\right]+[\frac{\nu\left(\frac{1}{\nu^2}+\frac{1}{8}\psi_2\left(\frac{\nu}{2}\right)\right)}{2\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)}]\right)\Bigg]$$

$$\frac{4D_0}{\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)}+\left(\frac{16\left(-\frac{1}{\nu}+\frac{1}{4}\psi_1\left(\frac{\nu}{2}\right)\right)}{\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)^3}\left[\begin{matrix}\left(\frac{\left(5-\nu\psi_1\left(\frac{\nu}{2}\right)\right)}{2}+\frac{1}{2}\left(\frac{\left[\left(-\nu\psi_2\left(\frac{\nu}{2}\right)-2\psi_1\left(\frac{\nu}{2}\right)\right)D_0-1\right]\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)-\left[\left(9-2\nu\psi_1\left(\frac{\nu}{2}\right)\right)D_0-\nu\right]}{\left(\psi_1\left(\frac{\nu}{2}\right)+\frac{\nu}{2}\psi_2\left(\frac{\nu}{2}\right)\right)}\right)\right)\\ \left(\frac{1}{4D_0}-\frac{\nu}{4D_0^2}\right)\\ +\left(\nu\psi_1\left(\frac{\nu}{2}\right)-5\right)\\ \left(\left[\frac{1}{4}\left(-\frac{1}{\nu}+\frac{1}{4}\psi_1\left(\frac{\nu}{2}\right)\right)\right]+\left[\frac{\nu}{8D_0^2}\left(9-2\nu\psi_1\left(\frac{\nu}{2}\right)\right)\right]\right)\end{matrix}\right]\right)$$

$$+\frac{2\Big(\nu\psi_1\big(\frac{\nu}{2}\big)-4\Big)}{\nu^2\Big((ln2-3)+\psi\big(\frac{\nu}{2}\big)\Big)}\Big(\frac{1}{4D_0}\Big)=\frac{8\pi g\,\varpi_{12}^{(0)}}{c_{light}^4} \qquad ...(4.12)$$

$$G_{21}^{(0)}=\frac{8\nu}{\Big(\nu\psi_1\big(\frac{\nu}{2}\big)-5\Big)^3}\Bigg[\Big(\Big[\Big(\psi_1\big(\frac{\nu}{2}\big)+\frac{\nu}{2}\psi_2\big(\frac{\nu}{2}\big)\Big)+\Big(\frac{1}{\nu^2}+\frac{1}{4}\psi_2\big(\frac{\nu}{2}\big)\Big)D_0\Big]\Big(\nu\psi_1\big(\frac{\nu}{2}\big)-5\Big)-[\Big(\nu\psi_1\big(\frac{\nu}{2}\big)-5\Big)$$

$$+\Big(-\frac{1}{\nu}+\frac{1}{4}\psi_1\big(\frac{\nu}{2}\big)\Big)D_0]\Big(\psi_1\big(\frac{\nu}{2}\big)+\frac{\nu}{2}\psi_2\big(\frac{\nu}{2}\big)\Big)\Big))\frac{(-\frac{1}{\nu}+\frac{1}{4}\psi_1\big(\frac{\nu}{2}\big)-\frac{1}{4D_0})}{D_0}$$

$$+\frac{\Big(\nu\psi_1\big(\frac{\nu}{2}\big)-5\Big)}{64D_0^2}\Big(\Big[\Big(\frac{1}{\nu^2}+\frac{1}{8}\psi_2\big(\frac{\nu}{2}\big)\Big)D_0-2\Big(-\frac{1}{\nu}+\frac{1}{4}\psi_1\big(\frac{\nu}{2}\big)\Big)\Big]\Big)\Bigg]+\frac{2\Big(\nu\psi_1\big(\frac{\nu}{2}\big)-4\Big)}{\nu^2\Big((ln2-3)+\psi\big(\frac{\nu}{2}\big)\Big)}\Big(\frac{1}{4D_0}\Big)=\frac{8\pi g\,\varpi_{21}^{(0)}}{c_{light}^4} \qquad ...(4.13)$$

$$G_{22}^{(0)}=\frac{2}{\Big(\nu\psi_1\big(\frac{\nu}{2}\big)-5\Big)^3}[\frac{1}{D_0}\Bigg[\Big(\Big(\Big(-2\nu\psi_1\big(\frac{\nu}{2}\big)-\frac{\nu}{2}\psi_2\big(\frac{\nu}{2}\big)\Big)D_0-1)\Big(\nu\psi_1\big(\frac{\nu}{2}\big)-5\Big)-\Big(9-2\nu\psi_1\big(\frac{\nu}{2}\big)\Big)D_0-\nu)\Big(\psi_1\big(\frac{\nu}{2}\big)+\frac{\nu}{2}\psi_2\big(\frac{\nu}{2}\big)\Big)\Big)$$

$$+\Big[\Big(\nu\psi_1\big(\frac{\nu}{2}\big)-5\Big)\Big]\Big)\Big(-\frac{1}{4D_0}+\frac{\nu}{4D_0^2}\Big)+\frac{1}{4}\Big(\Big[\frac{9-2\nu\psi_1\big(\frac{\nu}{2}\big)}{D_0}\Big]-\Big[\Big(-\frac{1}{\nu}+\frac{1}{4}\psi_1\big(\frac{\nu}{2}\big)\Big)\Big]\Big)\Bigg]-2\nu$$

$$\Bigg[\Bigg(\Bigg(\frac{\Big(\nu\psi_1\big(\frac{\nu}{2}\big)-5\Big)+\nu\Big(\psi_1\big(\frac{\nu}{2}\big)+\frac{\nu}{2}\psi_2\big(\frac{\nu}{2}\big)\Big)}{2D_0^2}+\frac{[\Big(\nu\psi_1\big(\frac{\nu}{2}\big)-5\Big)\Big(2\psi_1\big(\frac{\nu}{2}\big)+\nu\psi_2\big(\frac{\nu}{2}\big)\Big)+(9-2\nu\psi_1\big(\frac{\nu}{2}\big)\Big(\psi_1\big(\frac{\nu}{2}\big)+\frac{\nu}{2}\psi_2\big(\frac{\nu}{2}\big)\Big)]}{2D_0}\Big(9-2\nu\psi_1\big(\frac{\nu}{2}\big)\Big)\Bigg)$$

$$-\Bigg[\frac{\Big(\nu\psi_1\big(\frac{\nu}{2}\big)-5\Big)}{2D_0^2}\Bigg]\Bigg)\Big[-\frac{1}{\nu}+\frac{1}{4}\psi_1\big(\frac{\nu}{2}\big)-\frac{1}{4D_0}\Big]+\Big(\nu\psi_1\big(\frac{\nu}{2}\big)-5\Big)\frac{(1)}{32D_0^2}\Big(\nu\psi_2\big(\frac{\nu}{2}\big)+2\psi_1\big(\frac{\nu}{2}\big)\Big)\Bigg]$$

$$-\frac{\Big(\nu\psi_1\big(\frac{\nu}{2}\big)-4\Big)}{2\nu\Big((ln2-3)+\psi\big(\frac{\nu}{2}\big)\Big)D_0^2}=\frac{8\pi g\,\varpi_{21}^{(0)}}{c_{light}^4} \qquad ...(4.14)$$

Proof

Engaging the obtained results from theorems 3.1 and 4.1 together with the derivation noted in (4.13) and fully obtained in detail (Mageed, 2024), the proofs are immediate.

The following theorem is significant as it explains that based on Equation (2.8), which links Ricci scalar, Ricci curvature tensor, the Einsteinnian curvature **spacetime** tensor and the stress energy tensor,

$$G_{ij} = R_{ij}^{(\alpha)} - \frac{\mathcal{R}}{2}g_{ij} = \frac{8\pi g\varpi_{ij}}{c_{light}^4} \text{ (c.f., (2.8))}$$

If $\mathcal{R} = 0$,then

$$G_{ij} = R_{ij}^{(\alpha)} = \frac{8\pi g\varpi_{ij}}{c_{light}^4} \qquad ...(4.15)$$

(2.15) translates as Ricci scalar approaches zero, both Einesteinnian curvature **spacetime** and Ricci Tensors are equal.

Also,the Stress Energy tensor is proportional to both with a factor of $\frac{c_{light}^4}{8\pi g}$. Moreover, a value of $\mathcal{R} = 0$ means that volumes are invariant. But note that these are volumes of open sets in the manifold. So, if the underlying manifold is a

spacetime, the volumes are not spatial volumes, but rather have some dimension in time as well. A value of $\mathcal{R} = 0$, just means that some dimensions may stretch, and others may compress, but they do in such a way so that the overall volume does not change. (You can make an analogy with solid mechanics and the Poisson ratio). The Ricci tensor tells you how volumes change, and a Ricci scalar of zero means that volumes do not change. The other physical interpretation of a zero Ricci scalar is that in this important special case, GBM manifold will be a flat manifold. Furthermore, it translates to vacuum **spacetimes**.

4.2. Zeros of Ricci Scalar of GBM Manifold

Theorem 2.3: The zeros of Ricci scalar, $\mathcal{R}$ (c.f., (1.1)) are characterized by the path equation:

$\Gamma\left(\frac{\nu}{2}\right) = \Theta_2(\Theta_1\nu^4)^{\left(\frac{\nu}{2}\right)}e^{-\nu}$, provided that θ_1 and θ_2 are any two non-zero real constants. ...(4.16)

Proof

Let $\mathcal{R} = 0$.Then,it follows that

$$\nu\psi_1\left(\frac{\nu}{2}\right) = 4, \text{ or } \psi_1\left(\frac{\nu}{2}\right) = \frac{4}{\nu} \quad ...(4.17)$$

Define $x = \frac{\nu}{2}$.This transforms (4.17) to

$$d[\frac{d}{dx}(ln\Gamma(x))] = \frac{4dx}{x} \quad ...(4.18)$$

Integration solves (2.18) to

$$\left[\frac{d}{dx}(ln\Gamma(x))\right] = 4lnx + ln\varphi_1 = ln(\varphi_1 x^4) \quad ...(4.19)$$

Therefore,

$d(ln\Gamma(x)) = ln(\varphi_1 x^4)dx$, which that the closed form solution is determined by:

$$ln\Gamma(x) = \int ln(\varphi_1 x^4)dx = ln\varphi_1 \int dx + 4\int lnxdx = xln\varphi_1 + 4xlnx - 4x + ln\Theta_2 \quad ...(4.20)$$

Clearly, it follows that

$$e^{(xln\,\varphi_1 + 4xlnx - 4x + ln\varphi_2)} = \Theta_2 e^{-4x}\varphi_1^x(x^4)^x = \Theta_2 e^{-4x}\,(\varphi_1 x^4)^x \quad ...(4.21)$$

$x = \frac{\nu}{2}$ re-writes (4.21) to the compact form solution

$\Gamma\left(\frac{\nu}{2}\right) = \Theta_2(\Theta_1\nu^4)^{\left(\frac{\nu}{2}\right)}e^{-\nu}$, provided that θ_1 and θ_2 are any two non -zero real constants, $\Theta_1 = \frac{\varphi_1}{16}$ (c.f., (4.16))

The following theorem is the condition for which Ricci scalar of (1.1) is infinite A value of $\mathcal{R} \to \infty$, just means that *s* dimensions are stretching significantly large enough so that the overall volume changes.

4.3. Infinite Values of Ricci Scalar, $\mathcal{R}$ of GBM Manifold

Theorem 2.4: The infinite values of Ricci scalar, $\mathcal{R}$ (c.f., (1.1)) are satisfied by the paths:

$\nu = 0$ or $\Gamma\left(\frac{\nu}{2}\right) = \Theta_3 e^{(3-ln2)\nu}$, provided that θ_3 is any arbitrary non-zero real constant. ...(4.22)

Proof

Let $\mathcal{R} \to \infty$. Then, it follows that

$$\nu^2\left((ln2 - 3) + \psi\left(\frac{\nu}{2}\right)\right) = 0 \quad ...(4.23)$$

(4.23) holds if either:

$\nu = 0$ (c.f., (4.22))

or

$$\left((ln2-3)+\psi\left(\frac{\nu}{2}\right)\right)=0 \quad ...(4.24)$$

Define $y=\frac{\nu}{2}$. This transforms (4.24) to

$$\left[\frac{d}{dx}\left(ln\Gamma(x)\right)\right]=2(3-ln2) \quad ...(4.25)$$

Integration solves (4.25) to

$$ln\Gamma(x)=2(3-ln2)x+ln\,\Theta_3 \quad ...(4.26)$$

Therefore,

$\Gamma(x)=\Theta_3 e^{2(3-ln2)x}=\Theta_3 e^{(3-ln2)\nu}$ (c.f., (4.22))

5. Conclusion and Future Work

The current letter presents a breakthrough in revealing statistical info-geometric relativization of the GBM manifold. New avenues of future work involve the development of statistical info-geometric relativization of stable queueing systems and time-dependent queueing systems. This development will revolutionize classical queueing theory by analyzing as well as visualizing the stability dynamics of both stable and time dependent queueing systems with the help of the statistical info-geometric relativization techniques.

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