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Mathematical Modeling on the Rate of Transport of Pollutant Colloids Through Oscillating Flow in River Channels

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Abstract

Pollutants' particle transport from point sources through flow channels of municipal rivers have been implicated as a potential spread of the hazardous chemical, biological and physical pollution. The unsteady characteristic of an oscillatory laminar flow of water (Newtonian fluid) with uniform distribution of pollutant particles moving in a river flow channel, has been analytically investigated. The governing equations are composed of the continuity equations and the Navier-stokes equations. The oscillating flow is described by setting one side of boundaries to be a periodic function. The method of analytical expression is presented as an efficient alternative method to the high-performance computing resources of numerical methods whose complexity, availability and requirements delimit their usage. The solution is obtained in the form of a Bessel series. The effects of pollutant particles are described by three parameters: the mass concentration (φ) of the pollutant particles, their frequency (ω) of flow in river channel, and the velocity-lagging time (τ) that measures the rate at which the velocity of the pollutant particles adjusts onto the velocity of the clean water and this depends upon the size of the individual pollutant particles. At different times within the river flow channel the velocity profiles, wall shear stress, and flow rate are graphically represented due to the effects of pollutant parameters (φ) and (τ) and the results obtained are compared with those of the clean water particles.

Keywords: *Pollutant, Colloid, Oscillating flow, Mass concentration, Delayed time, Frequency, Number density*

1. Introduction

Municipal wastewater (or sewage) consists mostly of black and gray waters. It is a complex matrix containing inorganic, biological and organic compounds that can be found either in solution or suspended particles. These particles can be divided into categories based on their size: colloidal 0.001 – 1 μm), dissolved (<0.001 μm), settleable (<102 μm), supra-colloidal (1 – 102 μm) (Dulekgurgen *et al.*, 2006). Most of the soluble part consists mainly of inorganic compounds, whereas suspended particulate consists of organic material.

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The suspended particles are termed as pollutant colloids. They arise from a combination of industrial, agricultural and natural sources of pollution including soils and sands, and micro flora (Amir Hatamkhani and Ali Maridi, 2021; Moridi *et al.*, 2018).

In this paper, the effects of unsteady phenomena are studied together with suspended pollutant particles and clean water particles on oscillating flow of water through a stretching river flow channel (Ombaki and Kerongo, 2022). The river channel is considered to have the geometry of a pipe and hence we consider the basic equations in cylindrical co-ordinates. Intuitively, a viscous compressible pollutant water bounded by circular tube is executing simple harmonic oscillations with a frequency ω in river channel of radius $R = r_0$. The oscillating waterflow columns define particle trajectories, their transportation and final locations (Kongunan and Pholuang, 2012).

The following assumptions were considered to formulate the mathematical model:

i) The pollutant particles are spherical in shape and are uniformly distributed in the river flow channel.

ii) The pollutant number density N, and therefore the pollutant concentration (φ), are constant throughout the river flow channel.

iii) The flow is unsteady (under the assumption of constant pressure gradient).

iv) The proportion $\left(\frac{\rho_c}{\rho_p}\right)$ (where, ρ_c and ρ_p are the density of clean water and polluted water, respectively.) is assumed to be very small so as to neglect the buoyancy force.

In this paper, we investigate the effects of wall shear stress, flow rate and velocity of clean water in the presence of pollutant particles for varying radial coordinates and for varying time considering the pollutant parameters φ and τ. By incorporating the effects of suspended pollutant particles (colloids) transported in clean water (Mojtaba *et al.*, 2016), the analytical solution of the basic developed model equations are obtained. The analytical solutions are computed numerically and the results are presented in graphs.

2. Model Development

We assume that the pollutant particles in the polluted water and the clean water particles flow through the symmetric curve of semi-circular trough of river channel.

Denote: velocity of clean water, $U_c = U_c(x, y, z)$,

velocity of pollutant water, $U_p = U_p(x, y, z)$.

The flow is considered only in the x-axis. The equation of motion in the river channel is given by

$$\frac{\partial U_c^*}{\partial t^*} = -\frac{1}{\rho}\frac{dp^*}{dx} + \beta_0\left(\frac{d^2U_c^*}{dr^2} + \frac{1}{r}\frac{dU_c^*}{dr}\right) + \frac{Kn}{\rho}\left(U_p^* - U_c^*\right) \quad ...(1)$$

$$M\frac{\partial U_p^*}{\partial t^*} = K\left(U_c^* - U_p^*\right) \quad ...(2)$$

The initial and boundary conditions to solve Equations (1) and (2) are set as

$$U_c^* = \frac{c}{4\omega L}\left(1 - \frac{r}{R_O}\right), \quad ...(3)$$

$$\frac{\partial U_c^*}{\partial r} = 0,\ at\ r = 0;\ U_p^* = U_0^*\left(\frac{\sin\left(t^*\right)}{\omega L^2}\right)\ at\ r_0 = 1, \quad ...(4)$$

where, p^* is pressure gradient (assumed to be constant); β_c is kinematic viscosity of the clean water; M is the mass of a pollutant particles; K is the stokes' drag coefficient given as $K = 6\pi r\mu$ for spherical particles of radius r; n_0 is number of pollutant particles; μ is the viscosity of the clean water; and t^* is the time of particle flow.

The Equations of motion (3) and (4) are made dimensionless using the following parameterization (Saffman, 1961):

$$\lambda = \frac{r}{R_O};\ Y = \frac{x}{L};\ p = \frac{p^* L\omega^2}{\rho};\ t = \frac{t^*}{L^2\omega};\ U_c = U_c^* l\omega;\ \tau = \frac{m}{\omega k L^2};$$

$$U_p = U_p^* L\omega;\ Q = \frac{kn_0 L^2\omega^2}{\rho_c};\ \phi = Q\tau = \frac{kn_0}{\rho_c},$$

where,

τ = delay in mixing process; ϕ = mass concentration of the pollutant particles;

ω = frequency in the water flow channel; R = axial radius of the river channel; L = length of the channel.

Substituting the above non-dimensional quantities into Equations of motion (3) and (4) yields

$$\frac{\partial U_c}{\partial t} = -\frac{\partial p}{\partial y} + \frac{L^2}{R_o^2}\left(\frac{\partial^2 U_c}{\partial R^2} + \frac{1}{R}\frac{\partial U_c}{\partial R}\right) + \alpha\left(U_p - U_c\right) \quad \text{...(5)}$$

$$\tau\frac{\partial U_c}{\partial t} = k\left(U_c - U_p\right) \quad \text{...(6)}$$

The non-dimensional initial and boundary conditions are

$$U_p = \frac{c}{4}\left(1-\lambda^2\right),\ t = 0 \quad \text{...(7)}$$

$$\frac{\partial U_c}{\partial \lambda} = 0,\ at\ \lambda = 0;\ U_c = U_0 \sin(t)\ at\ \lambda_0 = 1. \quad \text{...(8)}$$

We consider that the pressure gradient $-\frac{\partial p}{\partial y}$ is constant such that

$$-\frac{\partial p}{\partial y} = c \quad \text{...(9)}$$

3. Method of Solution

We observe that the method of separation of variables fails to solve the model Equation (5) because the solution obtained through this method will not satisfy the set initial condition (7). Therefore, we proceed to find the transient solutions of Equation (5) by decomposing the velocity regime into unsteady part and steady parts such that

$$U_c(\lambda, t) = U_{cs}(\lambda) + U_{ct}(\lambda, t) \quad \text{...(10)}$$

$$U_P(\lambda, t) = U_{Ps}(\lambda) + U_{Pt}(\lambda, t), \quad \text{...(11)}$$

where, $U_{ct}(\lambda, t)$ and $U_{pt}(\lambda, t)$ are the transient (unsteady) parts while $U_{cs}(\lambda)$ and $U_{ps}(\lambda)$ are the steady parts of the velocity regime.

Inserting Equations (10) and (11) in Equations (5) and (6) and separating the steady part, we obtain

$$\frac{\partial^2 U_{cs}}{\partial \lambda^2} + \frac{1}{\lambda}\frac{\partial U_{cs}}{\partial \lambda} + c = 0 \quad \text{...(12)}$$

The set of boundary conditions to solve Equation (12) is

$$\frac{\partial U_{cs}}{\partial \lambda} = 0,\ at\ \lambda = 0;\ U_{cs} = U_0 \sin(t)\ at\ \lambda_0 = 1. \quad \text{...(13)}$$

Then the solution of Equation (12) satisfying the first condition in (13) is obtained as

$$U_{cs} = \frac{c}{4}\left(1-\lambda^2\right) + U_0 \sin\left(t\right). \qquad ...(14)$$

Again, the transient state of clean water is obtained by solving the unsteady Equations (5) and (6) using Equations (10) and (11) to obtain

$$\frac{\partial U_{ct}}{\partial t} = \frac{\partial^2 U_{ct}}{\partial \lambda^2} + \frac{1}{\lambda}\frac{\partial U_{ct}}{\partial \lambda} + \eta\left(U_{pt} - U_{ct}\right) \qquad ...(15)$$

3.1. Solving Equation (15) for the Flow of Clean Water Particles

We adopt the Laplace transforms into Equation (15) such that

$$V_{ct}\left(\lambda, S\right) = \mathcal{L}\left(\left(U_{ct}\left(\lambda, t\right)\right)\right) = \int_0^\infty U_{ct}\left(\lambda, t\right) e^{-St} dt \qquad ...(16)$$

$$V_{pt}\left(\lambda, S\right) = \mathcal{L}\left(\left(U_{pt}\left(\lambda, t\right)\right)\right) = \int_0^\infty U_{pt}\left(\lambda, t\right) e^{-St} dt \qquad ...(17)$$

where, S is conventionally a Laplace parameter and $\mathcal{L}$ is a Laplace transform. V_{ct} and V_{pt} are the Laplace transforms of U_{ct} and U_{pt}, respectively. The initial and boundary conditions associated with these Laplace transformations are set as:

Initial conditions,

$$V_{ct}\left(\lambda\right) = \frac{c}{4}\left(1-\lambda^2\right),\ t = 0. \qquad ...(18)$$

Boundary conditions,

$$\frac{\partial V_{pt}}{\partial \lambda} = 0,\ at\ \lambda = 0;\ V_{ct}\left(\lambda, t\right) = U_0 \sin\left(t\right)\ at\ \lambda_0 = 1. \qquad ...(19)$$

Applying Laplace transform to Equation (15) and (6) ad inserting conditions (18) and (19) results to an ordinary differential equation,

$$\frac{d^2 V_{ct}}{d\lambda^2} + \frac{1}{\lambda}\frac{dV_{ct}}{dt} - \left(\alpha + S\right)V_{ct} + \alpha V_{pt} = \frac{c}{4}\left(1-\lambda^2\right), \qquad ...(20)$$

$$V_{ct} = \left(\tau S + 1\right)V_{pt} - \frac{c\tau}{4}\left(1-\lambda^2\right). \qquad ...(21)$$

Eliminating V_{pt} between Equations (20) and (21) generates,

$$-\frac{D}{4}\left(1-\lambda^2\right) = \frac{d^2 V_{ct}}{d\lambda^2} + \frac{1}{\lambda}\frac{dV_{ct}}{dt} - \Omega V_{ct}, \qquad ...(22)$$

where, $\Omega = \dfrac{S^2\tau + \left(\alpha\tau + 1\right)S}{S\tau + 1}$, $D = \dfrac{c\left(\tau S - \alpha\tau + 1\right)S}{S\tau + 1}$.

To solve Equation (22), we apply Hankel transform of the form

$$\overline{V_{ct}} = V_{ct}\left(\lambda\right) = \int_0^1 V_{ct}\left(\lambda\right)\ \lambda J_0\left(\varepsilon\lambda\right) d\lambda. \qquad ...(23)$$

where, $J_0\left(\varepsilon_n\lambda\right)$ is a kernel implying Bessel function of order zero and $J_0(\varepsilon_n)$ is the n^{th} root of the Bessel equation of order zero.

Applying Equation (23) to solve Equation (22) yields,

$$\overline{V_{ct}} = \left[\frac{DJ_1(\varepsilon_n)}{2(\varepsilon_n^3)}\right]\left[\frac{1}{\Omega+\varepsilon_n^2}\right] - \frac{U_0\varepsilon_n J_0'(\varepsilon_n)}{(1+S^2)(\Omega+\varepsilon_n^2)} \quad ...(24)$$

Now we take the inverse Hankel transform on Equation (24) to give

$$\mathcal{L}U_c = 2\sum_{n=1}^{5}\left(\frac{D}{2(\varepsilon_n^3)}\frac{1}{(\Omega+\varepsilon_n^2)} + \frac{U_0\varepsilon_n}{(1+S^2)(\Omega+\varepsilon_n^2)}\right)\frac{J_0(\varepsilon_n\lambda)}{J_1(\varepsilon_n)} \quad ...(25)$$

Considering that for clean water $\tau = 0, f = 0, D = c, \Omega = S$ and substituting these values in Equation (25) results to

$$\mathcal{L}U_c = 2\sum_{n=1}^{5}\left(\frac{c}{2\varepsilon_n^3}\right)\left(\frac{1}{(2\varepsilon_n^3+S)} + \frac{U_0\varepsilon_n}{(1+S^2)(S+\varepsilon_n^2)}\right)\frac{J_0(\varepsilon_n\lambda)}{J_1(\varepsilon_n)} \quad ...(26)$$

The inverse Laplace transform of (26) yields,

$$U_{ct}(\lambda, t) = 2\sum_{n=1}^{5}\left(\frac{ce^{\varepsilon_n^2 t}}{2\varepsilon_n^3}\right) + \left(\frac{1}{(2\varepsilon_n^3+S)} + U_0\varepsilon_n\left(\frac{-\cos t + \varepsilon_n^2\sin t + e^{\varepsilon_n^2 t}}{(1+\varepsilon_n^4)}\right)\right)\frac{J_0(\varepsilon_n\lambda)}{J_1(\varepsilon_n)} \quad ...(27)$$

Using Equations (14) and (27) in Equations (10) and (11), we obtain the required velocity of clean water in the form,

$$U_c(\lambda, t) = \frac{c}{4}(1-\lambda^2) + U_0\sin(t) + 2\sum_{n=1}^{5}\left(\frac{ce^{\varepsilon_n^2 t}}{2\varepsilon_n^3}\right) + \left(\frac{1}{(2\varepsilon_n^3+S)} + U_0\varepsilon_n\left(\frac{-\cos t + \varepsilon_n^2\sin t + e^{\varepsilon_n^2 t}}{(1+\varepsilon_n^4)}\right)\right)\frac{J_0(\varepsilon_n\lambda)}{J_1(\varepsilon_n)} \quad ...(28)$$

3.2. Solving Equation (15) for the Flow of Pollutant Colloids

In this case we consider the case when $\tau \ll \frac{L}{V_{ct}}$ such that $V_{ct} = V_{pt}$ for disturbance over the length L or larger. Equation (6) becomes

$$U_c - U_p = \tau\frac{\partial U_c}{\partial t} \quad ...(29)$$

Inserting Equation (29) in Equation (5) leads to

$$\frac{\partial U_c}{\partial t} = -\frac{\partial p}{\partial y} + \left(\frac{\partial^2 U_c}{\partial\lambda^2} + \frac{1}{\lambda}\frac{\partial U_c}{\partial\lambda}\right) - \Omega\tau\frac{\partial U_c}{\partial t}$$

Or, $\frac{\partial U_p}{\partial t} + \Omega\tau\frac{\partial U_c}{\partial t} = -\frac{\partial p}{\partial y} + \left(\frac{\partial^2 U_c}{\partial\lambda^2} + \frac{1}{\lambda}\frac{\partial U_c}{\partial\lambda}\right)$

Or $(1+\alpha\tau)\frac{\partial U_c}{\partial t} = \frac{\partial^2 U_c}{\partial\lambda^2} + \frac{1}{\lambda}\frac{\partial U_c}{\partial\lambda}$ ...(30)

For the fine pollutant particles, $\alpha\tau = F$. Then,

$$(1+F)\frac{\partial U_c}{\partial t} = c + \left(\frac{\partial^2 U_c}{\partial\lambda^2} + \frac{1}{\lambda}\frac{\partial U_c}{\partial\lambda}\right) \quad ...(31)$$

Choosing $U_c = U_F$ for the fine pollutant particles gives

$$\left(1+F\right)\frac{\partial U_F}{\partial t} = c + \left(\frac{\partial^2 U_F}{\partial \lambda^2} + \frac{1}{\lambda}\frac{\partial U_F}{\partial \lambda}\right) \qquad ...(32)$$

The initial and boundary conditions to solve Equation (32) are set as:

Initial condition,

$$u_F = \frac{c}{4}\left(1-\lambda^2\right) + u_0 \sin\left(t\right) \qquad ...(33)$$

Boundary conditions,

$$\frac{\partial u_{Fs}}{\partial \lambda} = 0,\ at\ \lambda = 0;\ u_{Fs} = u_0 \sin\left(t\right)\ at\ \lambda_0 = 1 \qquad ...(34)$$

Decomposing the velocity profile of fine pollutant colloids into unsteady and steady parts we get

$$\frac{\partial^2 u_{Fs}}{\partial \lambda^2} + \frac{1}{\lambda}\frac{\partial u_{Fs}}{\partial \lambda} + c = 0 \qquad ...(35)$$

To solve Equation (35) for steady state, the boundary conditions are

$$\frac{\partial u_{Fs}}{\partial \lambda} = 0,\ at\ \lambda = 0;\ u_{Fs}\left(\lambda\right) = u_0 \sin\left(t\right)\ at\ \lambda = 1 \qquad ...(36)$$

Solution for Equation (35) projects to

$$u_{Fs}\left(\lambda\right) = \frac{c}{4}\left(1-\lambda^2\right) + u_0 \sin\left(t\right) \qquad ...(37)$$

From Equation (33) the unsteady state is

$$\left(1+F\right)\frac{\partial u_{Ft}}{\partial t} = \frac{\partial^2 u_{Ft}}{\partial \lambda^2} + \frac{1}{\lambda}\frac{\partial u_F}{\partial \lambda} \qquad ...(38)$$

From Equation (35) the initial and boundary conditions to solve Equation (35) are

$$u_{Fs}\left(\lambda\right) = u_{Ft}\left(\lambda\right) = \frac{c}{4}\left(1-\lambda^2\right) + u_0 \sin\left(t\right)\ at\ t = 0;\ \frac{\partial u_{Ft}}{\partial \lambda} = 0\ at\ \lambda = 1;$$

$$u_{Ft}\left(\lambda,\ t\right) = 0,\ at\ \lambda = 0 \qquad ...(39)$$

The Laplace transform for two variables λ, t for pollutant water is

$$U_{Ft}\left(\lambda,\ S\right) = \mathcal{L}\left[u_{Ft}\left(\lambda,\ t\right)\right] = \int_0^\infty e^{-St} u_{Ft}\left(\lambda,\ t\right) dt \qquad ...(40)$$

Thus,

$$\frac{\partial^2 u_{Ft}}{\partial \lambda^2} + \frac{1}{\lambda}\frac{\partial u_{Ft}}{\partial \lambda} - S\left(1+F\right)U_{Ft} + \frac{c}{4}\left(1-\lambda^2\right)\left(1+F\right) = 0 \qquad ...(41)$$

Again, we solve this equation using Hankel transform of the form

$$\overline{U_{Ft}} = \int_0^\infty u_{Ft}\left(\lambda,\ t\right)\lambda J_0\left(\varepsilon_n,\ \lambda\right) d\lambda \qquad ...(42)$$

Applying Equation (42) in Equation (34) yields

$$\overline{U_{Ft}} = \left(\frac{1}{S+\frac{\varepsilon_n^2}{1+F}}\right)\frac{cJ_1(\varepsilon_n)}{2\varepsilon_n^3} - \left(\frac{1}{\left(S^2+1\right)\left(S+\frac{\varepsilon_n^2}{1+F}\right)}\right)\frac{\varepsilon_n u_0 J_0'(\varepsilon_n)}{(1+F)} \quad ...(43)$$

We using the inverse Hankel transform of Equation (43) yields

$$\mathcal{L}\left[u_{Ft}(\lambda)\right] = \overline{U_{Ft}}(S) = 2\sum_{n=1}^{5}\left[\left(\frac{1}{\frac{\varepsilon_n^2}{1+F}}\right)\frac{c}{2\varepsilon_n^3} + \left(\frac{\varepsilon_n u_0}{(1+F)\left(S^2+1\right)\left(S+\frac{\varepsilon_n^2}{1+F}\right)}\right)\right]\frac{J_0(\varepsilon_n\lambda)}{J_1(\varepsilon_n)} \quad ...(44)$$

The inverse transform of Equation (44) is

$$u_F(\lambda, t) = 2\sum_{n=1}^{5}\left[\frac{c}{2\varepsilon_n^3}e^{\left(-\frac{\varepsilon_n^2}{1+F}\right)t} + \frac{\varepsilon_n u_0}{1+F}\left(\frac{-(1+F)^2\cos t + (1+F)\varepsilon_n^2(\sin t) + (1+F)\varepsilon_n^2 e^{\left(-\frac{\varepsilon_n^2}{1+F}\right)t}}{(1+F)^2\left(S^2+1\right)+\varepsilon_n^2}\right)\right]\frac{J_0(\varepsilon_n\lambda)}{J_1(\varepsilon_n)} \quad ...(45)$$

Summing up Equations (37) and (45) gives the desired velocity of the fine pollutant colloids. Thus,

$$u_F(\lambda, t) = u_{Fs}(\lambda) = \frac{c}{4}\left(1-\lambda^2\right) + u_0\sin(t)$$

$$+2\sum_{n=1}^{5}\left[\frac{c}{2\varepsilon_n^3}e^{\left(-\frac{\varepsilon_n^2}{1+F}\right)t} + \frac{\varepsilon_n u_0}{1+F}\left(\frac{-(1+F)^2\cos t + (1+F)\varepsilon_n^2(\sin t) + (1+F)\varepsilon_n^2 e^{\left(-\frac{\varepsilon_n^2}{1+F}\right)t}}{(1+F)^2\left(S^2+1\right)+\varepsilon_n^2}\right)\right]\frac{J_0(\varepsilon_n\lambda)}{J_1(\varepsilon_n)} \quad ...(46)$$

3.3. The Mass Flow Rate of Clean Water

Now, to find the quantitative dispersion of pollutant colloids on transport of water in a river channel, we calculate the mass flow rate and wall shear stress which is the skin friction.

The mass flow rate of clean water is given by

$$Q_w(t) = \frac{2\pi\gamma_w\lambda_0^2}{L}\int_0^1 (u_w, \lambda)\, d\lambda$$

$$= \frac{2\pi\gamma_w\lambda_0^2}{L}\int_0^1\left[\frac{c}{4}\left(1-\lambda^2\right) + u_0\sin(t) + 2\sum_{n=1}^{5}\left(\frac{ce^{-\varepsilon_n^2 t}}{2\varepsilon_n^3}\right) + \left(u_0\varepsilon_n\left(\frac{-\cos t + \varepsilon_n^2\sin t + e^{-\varepsilon_n^2 t}}{\left(1+\varepsilon_n^4\right)}\right)\right)\frac{J_0(\varepsilon_n\lambda)}{J_1(\varepsilon_n)}\right]\lambda\, d\lambda \quad ...(47)$$

We integrate Equation (47) to obtain

$$Q_w(t) = \frac{2\pi\gamma_w\lambda_0^2}{L}\left[\frac{c}{16} + \frac{u_0}{2}\sin(t) + 2\sum_{n=1}^{5}\left(\frac{ce^{\varepsilon_n^2 t}}{2\varepsilon_n^4}\right) + \left(u_0\left(\frac{-\cos t + \varepsilon_n^2\sin t + e^{-\varepsilon_n^2 t}}{\left(1+\varepsilon_n^4\right)}\right)\right)\right] \quad ...(48)$$

3.4. Mass Flow Rate for Pollutant Particles

Also, the mass flow rate of pollutant colloids is

$$Q_F(t)=\frac{2\pi\gamma_w\lambda_0^2}{L}\int_0^1(u_F,\ \lambda)\,d\lambda$$

$$=\frac{2\pi\gamma_w\lambda_0^2}{L}\int_0^1\left[\frac{c}{4}\left(1-\lambda^2\right)+u_0\sin(t)+2\sum_{n=1}^{5}\left(\frac{c}{2\varepsilon_n^3}e^{\left(-\frac{\varepsilon_n^2}{1+F}\right)t}\right)+\left(\varepsilon_n u_0\left(\frac{-(1+F)^2\cos t+\varepsilon_n^2(\sin t)+(1+F)\varepsilon_n^2 e^{\left(-\frac{\varepsilon_n^2}{1+F}\right)t}}{(1+F)^2+\varepsilon_n^4}\right)\right)\frac{J_0(\varepsilon_n\lambda)}{J_1(\varepsilon_n)}\right]\lambda d\lambda \quad \text{...(49)}$$

Integrating Equation (49) yields,

$$Q_F(t)=\frac{2\pi\gamma_w\lambda_0^2}{L}\left[\frac{c}{16}+\frac{u_0}{2}\sin(t)+2\sum_{n=1}^{5}\left(\frac{c}{2\varepsilon_n^4}e^{\left(-\frac{\varepsilon_n^2}{1+F}\right)t}\right)+u_0\left(\frac{-(1+F)^2\cos t+\varepsilon_n^2(\sin t)+e^{\left(-\frac{\varepsilon_n^2}{1+F}\right)t}}{(1+F)^2+\varepsilon_n^4}\right)\right] \quad \text{...(50)}$$

3.5. Wall Shear Stress

The skin friction (or wall shear stress) for clean water is

$$\tau_w(t)=\left[-\mu_w\rho\gamma_0^2(u_w)_\lambda\right]_{\lambda=1}$$

$$=-\mu_w\rho\gamma_0^2\left[\frac{c}{4}\left(1-\lambda^2\right)+u_0\sin(t)+2\sum_{n=1}^{5}\left(\frac{ce^{-\varepsilon_n^2 t}}{2\varepsilon_n^3}\right)+\left(u_0\varepsilon_n\left(\frac{-\cos t+\varepsilon_n^2\sin t+e^{-\varepsilon_n^2 t}}{\left(1+\varepsilon_n^4\right)}\right)\right)\frac{J_0(\varepsilon_n\lambda)}{J_1(\varepsilon_n)}\right]_{\lambda=1}$$

$$=-\mu_w\rho\gamma_0^2\left[\frac{c}{4}+2\sum_{n=1}^{5}\left(u_0\varepsilon_n^2\left(\frac{-\cos t+\varepsilon_n^2\sin t+e^{-\varepsilon_n^2 t}}{\left(1+\varepsilon_n^4\right)}\right)\right)-\frac{e^{-\varepsilon_n^2 t}}{2\varepsilon_n^2}\right] \quad \text{...(51)}$$

Skin friction for fine pollutant particles is obtained as in Equation (50) below.

$$\tau_F(t)=\left[-\mu_F\rho\gamma_0^2(u_F)_\lambda\right]_{\lambda=1}$$

$$=\mu_F\rho\gamma_0^2\left[\frac{c}{4}\left(1-\lambda^2\right)+u_0\sin(t)+2\sum_{n=1}^{5}\frac{c}{2\varepsilon_n^3}e^{\left(-\frac{\varepsilon_n^2}{1+F}\right)t}+\left(\varepsilon_n u_0\left(\frac{-(1+F)^2\cos t+\varepsilon_n^2(\sin t)+\varepsilon_n^2 e^{\left(-\frac{\varepsilon_n^2}{1+F}\right)t}}{(1+F)^2+\varepsilon_n^4}\right)\right)\frac{J_0(\varepsilon_n\lambda)}{J_1(\varepsilon_n)}\right]_\lambda$$

$$=\mu_F\rho\gamma_0^2\left[\frac{c}{2}+2\sum_{n=1}^{5}\left(u_0\varepsilon_n^2\left(\frac{-(1+F)\cos t+\varepsilon_n^2\sin t+(1+F)e^{\left(-\frac{\varepsilon_n^2}{1+F}\right)t}}{(1+F)^2+\varepsilon_n^2}\right)\right)-\frac{e^{-\varepsilon_n^2 t}}{2\varepsilon_n^2}\right] \quad \text{...(52)}$$

4. Graphical Results

The Figures 1 to 3 shown below were generated in Matlab software using Equations (48), (50) and (52), respectively. In the Figure 1, axial velocity of particle flow was plotted against axial radius of river channel for both colloidal and clean water at various flow time, *t*.

In the Figure 2, variation of flow rate Q with time *t* for different values of viscosity $n = \mu$, for $F = 0.4$ and $\tau = 0.2$ was displayed. In the Figure 3, variation of wall shear stress τ with axial position time *t* for different values of viscosity $n = \mu$, for $F = 0.4$ and $\tau = 0.2$ were also displayed.

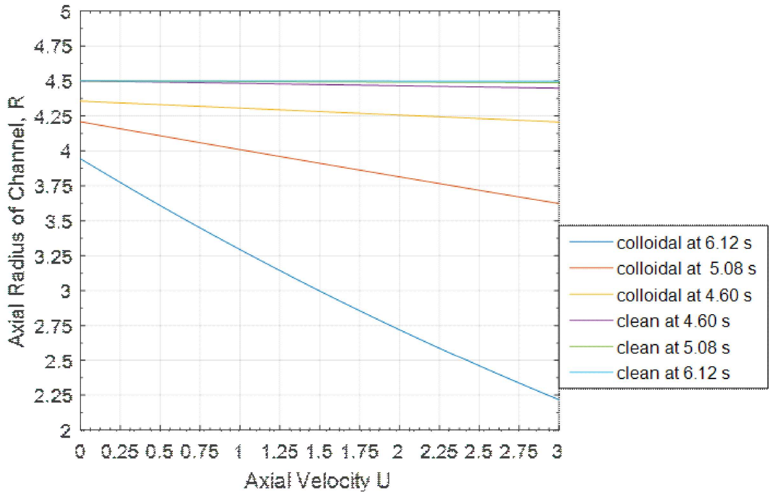

Figure 1: Variation of Axial Velocity *U* with Radial Position *R* for Different Values of Time *t* (for *F* = 0.4, τ = 0.2)

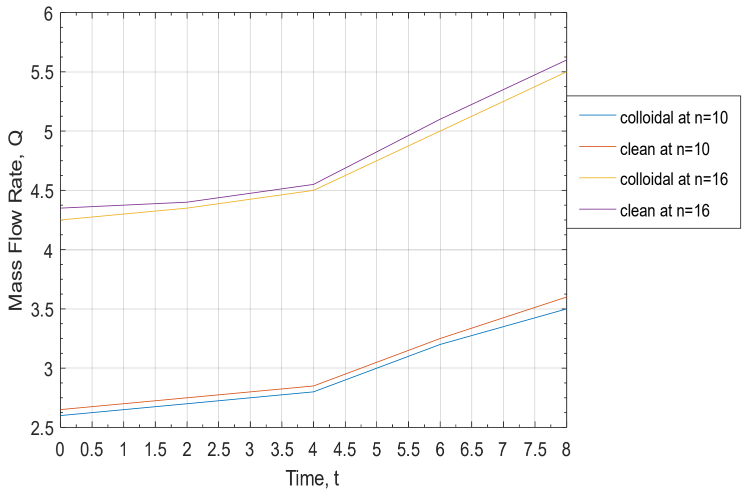

Figure 2: Variation of Flow Rate *Q* with Time *t* for Different Values of Viscosity *n* = *m*, (for *F* = 0.4, τ = 0.2)

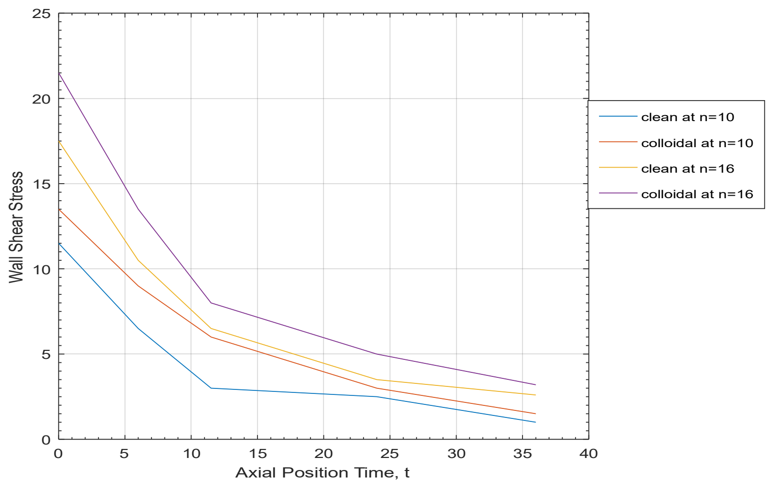

Figure 3: Variation of Wall Shear Stress *t* with Axial Position Time *t* for Different Values of Viscosity *n* = *m*, (for *F* = 0.4, τ = 0.2)

5. Conclusion

Wall shear stress, flow rate, and velocity have been mathematically presented as the main parameters that influence the pollutant particle transport through a river flow channel. It has been realized that these parameters depend more on the frequency and the mass concentration (φ) of the pollutant particles than on the size of these particles. The changes in the wall shear stress, flow rate and velocity of pollutant particles with varying time t due to the pollutant parameters F, τ and frequency ω are computed. The results are depicted in the graphs.

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