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Meromorphic Continuation of Global Zeta Function for Number Fields

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Abstract

In the paper, we shall establish the existence of a meromorphic continuation of the Global Zeta Function $\zeta(f, X)$ of a Global Number Field K and also deduce the functional equation for the same, using different properties of the idéle class group C_K^1 of a global field K extensively defined using basic notions of Adéles ($\mathbb{A}_K$) and Idéles ($\mathbb{I}_K$) of K, and also evaluating Fourier Transforms of functions f on the space $S(\mathbb{A}_K)$ of Adélic Schwartz-Bruhat Functions. A brief overview of most of the concepts required to prove our desired result have been provided to the readers in the earlier sections of the text.

Keywords: *Adéles, Idéles, Global Zeta Function, Archimedean Valuations, Non Archimedean Valuations, Restricted Direct Products, Fourier Transforms, Riemann-Roch Theorem, Characters of a Group, Haar Measure, Schwartz-Bruhat Spaces, Adélic Schwartz-Bruhat Functions, Local Rings, Poisson Summation Formula, Meromorphic Continuation*

1. Adéles and Idéles: A Brief Introduction

First we introduce some important notions and terms which we shall use to describe the notion of Adéles and Idéles later on explicitly.

1.1 Restricted Direct Products

Consider $I := \{\nu\}$ to be any indexing set, and I_∞ be any fixed finite subset of I. Assuming that, we have a locally compact group G_ν (not neccessarily abelian), corresponding to every index $\nu \in I$, and also suppose we obtain a compact open (consequently closed also under the topology) group H_ν, relative to every such index $\nu \notin I_\infty$ having a subgroup structure corresponding to the same index ν. Thus, we have the following definition:

Definition 1.1.1: Restricted Direct Product

For any $\nu \notin I_\infty$, we define the restricted direct product of the group G_ν with respect to the subgroup H_ν as:

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$$G : \prod_{v \in I}' G_v = \left\{ (x_v)_v \mid x_v \in G_v \text{ with } x_v \in H_v \text{ for infinitely many } v \right\} \qquad ...(1.1)$$

Remark 1.1.2.

The topology on G is defined by fixing a neighborhood base for the identity element which consists of sets of the form $\prod_v N_v$, N_v being a neighborhood of 1 in G_v such that, $N_v = H_v$ for infinitely many v. Important to mention that, this topology is not the same as the product topology.

The restricted direct product G defined in Definition (1.1.1) satisfies some important portant properties:

Proposition 1.1.3.

The following holds for a restricted direct product G of G_v with respect to H_v defined in Definition (1.1.1):

- G is locally compact.
- Any subgroup $Y \subseteq G$ has a compact closure $\Leftrightarrow$ For some family $\{K_v\}_v$ of compact subsets such that, $K_v \subseteq G_v$, and $K_v = H_v$ for infinitely many indices v, we then have, $Y \subseteq \prod_v K_v$

1.2. Characters of Restricted Direct Products

We start with the definition of the character of a group:

Definition 1.2.1: Characters of a Group

A character χ of a topological group G is a continuous homomorphism $G \to \mathbb{C}^*$.

A priori using the notions of G defined in Definition (1.1.1), we have the following result:

Lemma 1.2.2.

For $\chi \in \mathrm{Hom}_{cont.}(G, \mathbb{C}^*)$, χ is trivial on all but finitely many H_v. Precisely, $\chi(y_v) = 1$ for infinitely many v, $\forall\ y := (y_v)_v \in G$, and also,

$$\chi(y) = \prod_v \chi(y_v)$$

Using the statement of the above lemma, we establish the following important result we shall use later on:

Lemma 1.2.3.

Suppose that, $\chi_v \in \mathrm{Hom}_{cont.}(G_v, \mathbb{C}^*)$ for every v, and, $\chi_v|H_v = 1$ for infinitely many v. Then, $\chi = \prod_v \chi_v$, and, $\chi \in \mathrm{Hom}_{cont.}(G_v, \mathbb{C}^*)$.

Suppose we consider the Pontryagin Dual of the restricted direct product G defined as in Definition (1.1.1). Consider the dual groups $\hat{G}_v$ of G_v. Then the following theorem gives us the following relation between $\hat{G}$ and $\hat{G}_v$ as:

Theorem 1.2.4.

$$\hat{G} \cong \prod_v' \hat{G}_v \qquad ...(1.2)$$

where, the restricted direct product $\prod_v' \hat{G}_v$ is with respect to the subgroups $K(G_v, H_v)$, where,

$$K(G_v, H_v) := \{\chi_v \in \mathrm{Hom}_{cont.}(G_v, \mathbb{C}^*) \mid \chi_v|H_v = 1\} \qquad ...(1.3)$$

Next, we define measures on G and $\hat{G}$.

1.3. Measures on Restricted Direct Products and their Duals

Proposition 1.3.1.

Assume $G = \prod_{v}' G_v$ to be the restricted direct product of locally compact groups G_v with respect to the family of compact subgroups $H_v \subseteq G_v$ for every $v \notin I_\infty$, we define the (left) Haar Measure on G_v to be dg_v and normalize it using the condition,

$$\int_{H_v} dg_v = 1, \text{ for almost all } v \notin I_\infty.$$

Then, there exists a Haar Measure dg on G sucth that, the restriction dg_S of dg to the group,

$$G_S = \prod_{v \in S} G_v \times \prod_{v \notin S} H_v$$

For every finite set $S\left(\supseteq I_\infty\right)$ of indices, is exactly equal to the product measure on G_S, and moreover, the Haar Measure dg on G is unique.

By the virtue of the above theorem, we can conclude that,

$$dg = \prod_{v} dg_v \text{ is well-defined} \quad ...(1.4)$$

And, dg is the (left) Haar Measure on G. This is also called the measure induced by the factor measures (precisely dg_v for every v).

Following result reflects some of the important properties of the left Haar Measure dg on G:

Proposition 1.3.2.

The following holds true for the (left) haar Measure on a restricted direct product G of locally compact groups G_v with respect to the compact subgroups H_v:

(i) Suppose f is integrable on G. Then,

$$\int_G f(g)\,dg = lim_S \int_{G_S} f(g_S)\,dg_S$$

S being any finite set of indices containing I_∞.

If f is assumed only to be continuous, then the above holds true provided that, the integral can take values at infinity.

(ii) Given any finite set of indices $S_0\left(\supseteq I_\infty\right)$ and all such v such that, $V\,ol(H_v, dg_v) \neq 1$, suppose we have a continuous integrable function f_v on G_v corresponding to each such index v such that,

$$f_v \,|_{H_v} = 1, \;\forall\, v \notin S_0$$

Then the relation $f(g) = \prod_{v} f_v(g_v), \;\forall\; g := (g_v)_v \in G$ is well-defined and continuous on G. Suppose S be any finite set of indices (S can be S_0), then,

$$\int_{G_S} f(g_S)\,dg_S = \prod_{v \in S}\left(\int_{G_v} f_v(g_v)\,dg_v\right)$$

More generally,

$$\int_G f(g)\,dg = \prod_{v}\left(\int_{G_v} f_v(g_v)\,dg_v\right)$$

And, also

$$f \in L^1(G), \text{provided, } \prod_{\nu}\left(\int_{G_\nu} f_\nu(g_\nu)\,dg_\nu\right) < \infty$$

(iii) $\{f_\nu\}$ and f be mentioned a priori, such that, moreover, f_ν is the characteristic function for H_ν for almost all ν. Then, f is integrable, and, in case of abelian groups, the Fourier Transform of f, i.e., $\hat{f}$ is likewise integrable and,

$$\hat{f}_\nu(g) = \prod_{\nu} \hat{f}_\nu(g_\nu).$$

Now, Assuming the expression of dg in Equation (1.4), we normalize each dg_ν for every such index ν such that, $Vol(H_\nu) = 1$ for almost all ν. And, suppose,

$d\chi_\nu = d\hat{g}_\nu$, for every ν.

where, $d\chi_\nu$ denotes the dual measure of dg_ν on $\hat{G}_\nu$ for every ν.

Hence for each ν and $f \in L^1(G_\nu)$, we have, using description of Forurier Transform,

$$\hat{f}_\nu(\chi_\nu) = \int_{G_\nu} f_\nu(g_\nu)\overline{\chi_\nu}(g_\nu)\,dg_\nu$$

In case of f_ν being the characteristic function of H_ν, hence being integrable and of positive type on G_ν for every ν, we deduce using orthogonality relations that,

$$\hat{f}_\nu(\chi_\nu) = \int_{H_\nu} \chi_\nu(g_\nu)\,dg_\nu = \begin{cases} Vol(H_\nu), & \text{if } \chi_\nu \mid H_\nu = 1, \\ 0, & \text{otherwise.} \end{cases} \quad ...(1.5)$$

Consequently, the group,

$$H_\nu^* = \left\{\chi_\nu \in Hom_{cont.}\left(G_\nu, C^*\right) \mid \quad \chi_\nu \mid H_\nu = 1\right\} = K\left(G_\nu, H_\nu\right), \text{using (1.3)} \quad ...(1.6)$$

is a subgroup of $\hat{G}_\nu$.

Therefore, applying Proposition 1.3.2.(ii), we thus obtain the Fourier Inversion Formula stated as:

Theorem 1.3.4: Fourier Inversion Formula

We have, for every $f \in V^1(G)$ (Space of L^1 functions in $V(G)$, where $V(G)$ denotes the complex span of continuous functions on G of positive type),

$$f(g) = \int_{\hat{G}} \hat{f}(\chi)\chi(g)\,d\chi \quad ...(1.7)$$

where, $\hat{G}$ is the Pontryagin Dual Group of G and $d\chi$ is the dual measure on $\hat{G}$ to the Haar Measure dg on G.

Using the Fourier Inversion formula and the Proposition (1.3.2), we conclude that,

$$Vol(H_\nu)\cdot Vol(H_\nu^*) = 1$$

where $Vol(H_\nu)$ is relative to the measure dg_ν, and $Vol(H_\nu^*)$ is relative to the measure $d\chi_\nu$; such that, $Vol(H_\nu^*) = 1$ for almost all ν, and, $d\chi = \hat{dg}$ as mentioned earlier.

1.4. Valuations on Number Fields

Definition 1.4.1: Absolute Value

An Absolute Value of a field K is a funtion, $|.| : K \to \mathbb{R}$ satisfying,

(i) $|x| \geq 0,\ \forall x \in K$, and, $|x| = 0 \Leftrightarrow x = 0$.

(ii) $|xy| = |x||y|,\ \forall x,\ y \in K$.

(iii) $|x+y| \leq |x| + |y|,\ \forall x,\ y \in K$ (triangle inequality).

Definition 1.4.2: Valuations

A valuation of a field *K* is a map, v : $K \leftarrow \mathbb{R} \cup \infty$ satisfying,

(i) $v(x) = \infty \Leftrightarrow x = 0$.

(ii) $v(xy) = v(x) + v(y),\ \forall x,\ y \in K$.

(iii) $v(x+y) \geq \min\cdot\{v(x),\ v(y)\},\ \forall x,\ y \in K$.

Definition 1.4.3: Non-Archimedean Valuation

The valuation v is defined to be non-Archimedean, if it satisfies all the three conditions mentioned in the Definition (1.4.2). Correspondingly, |.| is defined to be the non-Archimedean absolute value on *K*, if, $|n|$ is bounded for every $n \in \mathbb{N}$ on *K*.

Definition 1.4.4: Archimedean Valuation

The valuation v is defined to be Archimedean, if it does not satisfy the third condition, but satisfies the other two mentioned in the Definition (1.4.2). Correspondingly, |.| is defined to be the Archimedean absolute value on *K*, if, $|n|$ is bounded for every $n \in \mathbb{N}$ on *K*.

In other words, v is non-Archimedean if it is not Archimedean.

Clearly, from the above two definitions of valuations and absolute value on a field *K*, we can establish the following relation between them,

$$v(x) = \log(|x|),\ \forall x \in K.$$

where the log can be considered with respect to some prime p as the base, in case of p-adic valuations.

Example 1.4.5.

Consider the p-adic valuation and the p-adic absolute value on $\mathbb{Q}$ and $\mathbb{Z}$, which is non-Archimedean.

Proposition 1.4.6.

Every valuation on $\mathbb{Q}$ is either equivalent to the p-adic absolute value $|.|_p$ or the usual absolute value $|.|_\infty$.

Having defined all the tools neccessary, we finally introduce the notion of Adéeles and Idéles of a global field *K*.

1.5. Adéles and Idéles

Definition 1.5.1: Valuation Ring

Given a global field *K*, suppose we denote K_v as the completion of *K* at a non-Archimedean place v. Hence we define the Valuation Ring of K_v as,

$$O_v := \{x_v \in K_v \mid |x_v|_v \leq 1\}$$

$|.|_v$ denoting the non-Archimedean absolute value at a place v.

Remark 1.5.2.

$(K_v, +)$ is a locally compact additive group.

Example 1.5.3.

If K be an algebraic number field, then K_v shall either be $\mathbb{R}$, $\mathbb{C}$ or, a p-adic field.

For every finite place v each such K_v admits of a local ring of integers O_v as defined above; which is open and compact as a subgroup. Hence we have our definition:

Definition 1.5.4: Adéle Group

The Adéle Group $\mathbb{A}_K$ of a global field K is defined to be the restricted direct product of K_v over all v with respect to the subgroups O_v (v is finite), i.e.,

$$\mathbb{A}_K := \prod_v{}' K_v \qquad ...(1.8)$$

Again, for every finite place v, we consider the locally compact multiplicative groups $\left(K_v^*\right)$, then each such K_v^* admits of a local ring of units $O_v^\times$ (v is finite) which is open and compact as a subgroup. Thus we have our other definition:

Definition 1.5.5: Idéle Group

The Idéle Group $\mathbb{I}_K$ of a global field K is the group of units of the Adéle Group $\mathbb{A}_K$ and precisely is defined to be the restricted direct product of K_v^* over all v with respect to the subgroups $O_v^\times$ (v is finite), i.e.,

$$\mathbb{I}_K := \prod_v{}' K_v^* \qquad ...(1.9)$$

Remark 1.5.6.

From Definition (1.5.2), it is evident that, there exists a well-defined algebraic embedding,

$$K \rightarrow \mathbb{A}_K$$

$$x \mapsto (x, x, x, \ldots\ldots)$$

Remark 1.5.7.

From Definition (1.5.3), it is evident that, there exists a well-defined algebraic embedding,

$$K^* \rightarrow \mathbb{I}_K$$

$$x \mapsto (x, x, x, \ldots\ldots)$$

Remark 1.5.8.

The Adéle Group $\mathbb{A}_K$ admits of a ring structure ($\mathbb{A}_K$,+, .), and, a priori, by Definition (1.5.3), we have, as mentioned,

$$\mathbb{I}_K \cong \mathbb{A}_K^\times$$

Although this is not a topological embedding, since the toplology induced by $\mathbb{A}_K$ is coarser than the topology defined as the restricted direct product on $\mathbb{I}_K$.

Remark 1.5.9.

The field K as a group is discrete and cocompact subgroup of $\mathbb{A}_K$.

In terms of valuations, we can also give alternative definitions for Adéles and Idéles.

Definition 1.5.10: Adéle Group

Given a global field K, suppose for each finite place v, K_v denotes the completion of K at the place v, having O_v as the local valuation ring at the finite non-Archimedean place v defined as in Definition (1.5.1). Then we define the Adéle Group of K as,

$$\mathbb{A}_K := \prod_v{}' K_v = \left\{ x := (x_v)_v \mid x_v \in K_v \forall\, v \text{ and, } x_v \in O_v \text{ for infinitely many } v \right\} \quad ...(1.10)$$

$$= \left\{ x := (x_v)_v \mid x_v \in K_v \forall\, v \text{ and, } |x_v|_v \leq 1 \text{ for infinitely many } v \right\} \quad ...(1.11)$$

Definition 1.5.11: Idéle Group

Given a global field *K*, suppose for each finite place v, K_v denotes the completion of *K* at the place v, and K_v^* denotes the locally compact multiplicative group having $O_v^\times$ as the ring of units of the the local valuation ring O_v at the finite non-Archimedean place v defined as in Definition (1.5.1). Then we define the Id'ele Group of *K* as,

$$\mathbb{I}_K := \prod_v{}' K_v^* = \left\{ x := (x_v)_v \mid x_v \in K_v^* \forall\, v \text{ and, } x_v \in O_v^\times \text{ for infinitely many } v \right\} \quad ...(1.12)$$

$$= \left\{ x := (x_v)_v \mid x_v \in K_v^* \; \forall\, v \text{ and, } |x_v|_v \leq 1 \text{ for infinitely many } v \right\} \quad ...(1.13)$$

Definition 1.5.12: (Idéle Class Group)

Given a global field *K*, and $\mathbb{I}_K$ to be its Idéle Group, we define the idéle class group of *K* as,

$$C_K := \mathbb{I}_K / K^* \quad ...(1.14)$$

Next, we shall introduce the notion of absolute value on an Adéle $\mathbb{A}_K$ and on an Idéle $\mathbb{I}_K$ of *K*.

Definition 1.5.13.

Given a local field *k*, we define the normalized absolute value $|.|_k$ on *k* as:

- $|\cdot|_k = |\cdot|_\infty$, i.e., the usual absolute value if $k = \mathbb{R}$.
- $|z|_k = z \cdot \overline{z}$, $\forall\, z \in k$ if $k = \mathbb{C}$.
- For a non-Archimedean local field *K* with uniformizing papameter π, we have,

$$|\pi|_k = \frac{1}{q}, \text{ where, } q = |O_k / \pi \cdot \mathrm{O}_k|$$

Definition 1.5.14.

Given a global field *K*, we know that, for every finite place v, K_v is a local field which is also the completion of *K* at the place v hence we define the normalized absolute value, $|\cdot|_{\mathbb{A}_K} : \mathbb{A}_K \to \mathbb{R}_+^*$ in terms of the normalized absolute values $|.|_v$ on the completions K_v as:

$$|\cdot|_{\mathbb{A}_K} := \prod_v |x_v|_v, \; \forall \; x := (x_v)_v \in \mathbb{A}_K \quad ...(1.15)$$

Remark 1.5.15.

For every $x \in \mathbb{I}_K$, we have, $|x|_{\mathbb{A}_K} = 1$.

Remark 1.5.16.

Using the above definitions, we can conclude that, C_K is not coompact with respect this absolute value, although a priori, we have that, $\mathbb{A}_K/K$ is compact.

Definition 1.5.17: Idéle Class Group of Norm 1

Suppose *K* be an algebraic number field or a finitely generated function field in one variable over a finite field $\mathbb{F}_q$, where *q* is some power of prime. Then we define the norm 1 Idéle Group of *K* as,

$\mathbb{I}_K^1 := \ker\left(|\cdot|_{\mathbb{A}_K}\right)$, where, the map $|\cdot|_{\mathbb{A}_K}$ is defined in (1.15).

And, consequently, we give our definition of the Idéle Class Group of norm 1 as,

$$C_K^1 := \mathbb{I}_K^1 / K^* \qquad ...(1.16)$$

Remark 1.5.18.

C_K^1 is well-defined by the fact that, $K^* \mapsto \mathbb{I}_K^1$ (Using Artin's Product Formula)

Remark 1.5.19.

For any globbal field *K*, C_K^1 is compact.

Now that we have got the idea of the Adéle groups $\mathbb{A}_K$ and Idéle Groups $\mathbb{I}_K$ for a global field *K*, we intend to explicitly perform Fourier Transforms on a particular space $S(\mathbb{A}_K)$ of all the Adélic Schwartz-Bruhat Functions *f*.

1.6. Fourier Transforms on S($\mathbb{A}_K$)

Given a global field *K* and its completion K_ν at aplace ν and O_ν being the local ring of integers of K_ν at ν for every finite place ν; we can define corresponding Schwartz-Bruhat Spaces $S(K_\nu)$ of functions for every ν.

Using Definition (1.5.2), we define the space $S(\mathbb{A}_K)$ of Adélic Schwartz-Bruhat Functions as:

$$S(\mathbb{A}_K) := \otimes' S(K_\nu)$$

where $\otimes'$ denotes the restricted tensor product of the indivudual Schwartz-Bruhat Spaces $S(K_\nu)$ such that, $S(\mathbb{A}_K)$ has the following structure:

$$S(\mathbb{A}_K) := \left\{ f \in \otimes f_\nu \middle| f_\nu \in S(K_\nu) \forall \nu,\ f_\nu \middle| O_\nu = 1 \text{ for almost all } \nu \right\} \qquad ...(1.17)$$

Therefore,

$$f \in S(\mathbb{A}_K) \Rightarrow f(x) = \prod_\nu f_\nu(x_\nu),\ \forall x := (x_\nu)_\nu \in \mathbb{A}_K$$

And *f* is termed as the Adélic Schwartz-Bruhat Function.

Remark 1.6.1.

$S(\mathbb{A}_K)$ is dense in $L^2(\mathbb{A}_K)$, where $L^2(\mathbb{A}_K)$ is defined with respect to the Haar Measure on $\mathbb{A}_K$.

Definition 1.6.2: Fourier Transformation Formula

Given any $f \in S(\mathbb{A}_K)$, fixing a non-trivial continuous unitary character ψ on $\mathbb{A}_K$, we define the Adélic Fourier Transform of f as:

$$\hat{f}(y) := \int_{\mathbb{A}_K} f(x)\psi(xy)\,dx \qquad ...(1.18)$$

Where *dx* denotes the Haar Measure on $\mathbb{A}_K$ normalized by the self-dual measure for ψ.

Remark 1.6.3.

The map, $f \mapsto \hat{f}$ defines an automorphism on $S(\mathbb{A}_K)$ which extends to an isometry on $L^2(\mathbb{A}_K)$.

2. Global Zeta Function $\zeta(f, \chi)$

Definition 2.1.

Given any $\mathbb{C}^*$-valued character χ if the Idéle Class Group IK such that, $\chi \mid K^* = 1$; i.e., in other words, χ being a quasi-character of exponenet greater than 1; i.e., in other words, an Idéle Class Character; and for every $f \in S(\mathbb{A}_K)$, we define the Global Zeta Function of the field *K* as:

$$\zeta(f, \chi) := \int_{\mathbb{I}_K} f(x)\chi(x)d^*x \quad ...(2.1)$$

where, d^*x denotes the Haar Measure on $\mathbb{I}_K$, induced by the product measure $\prod_\nu d^*x_\nu$ on $\prod_\nu K_\nu^*$, for each non-Archimedean place ν.

Remark 2.2.

Note that, here, d^*x denotes the Haar Measure on K_ν^* for every non-Archimedean place ν, usually having the following representation:

$$d^*x_\nu := c_\nu \frac{dx_\nu}{|x_\nu|_\nu}$$

where, $c := (c_\nu)_\nu$ is some constant factor that is usually introduced in order to normalize d^*x.

$c := (c_\nu)_\nu$ can be evaluated as:

$$c_\nu = \frac{q_\nu}{q_\nu - 1}$$

where, $q_\nu := \mathcal{N}(\nu) = q^{deg(\nu)}$, where, $deg(\nu) := \left[\mathbb{F}_{q_\nu} : \mathbb{F}_\nu\right]$, at every finite place ν.

Such that, the Haar Measure on $O_\nu^\times$ (O_ν being the local ring of integers at every non-Archimedean place ν) shall be $= \sqrt{\left(\mathcal{N}\left(\mathcal{D}_\nu\right)^{-1}\right)} = \sqrt{q^{-d_\nu}}$, q be a prime.

where, $\mathcal{D}_\nu$ denotes the local different at every non-Archimedean place ν. In fact, $d_\nu = 0$ for all but finitely many non-Archimedean places ν.

In our main theorem explicitly stated and proved later, we shall establish the fact that, $\zeta(f, \chi)$ is normally convergent for $\sigma = Re(s) > 1$, and defines a holomorphic function in the region of its convergence, using rigorously, the fact that, χ has a representation, $\chi = \mu|.|^s$, where, μ is a unitary character of the Idéle Group $\mathbb{I}_K$.

3. Riemann-Roch Theorem

In this section, we shall state and prove one of the most unique and important theorems in the field of Harmonic Analysis on Adélic Groups which shall use to prove our main theorem. First, let us mention the Poisson Summation Formula in order to prove the Riemann-Roch Theorem.

Theorem 3.1: Poisson Summation Formula

Consider $f \in S(\mathbb{A}_K)$, i.e., f satisfies the following conditions,

(i) $f \in L^1(\mathbb{A}_K)$, and, f is continuous.

(ii) $\sum_{\gamma \in K} f(z(y+\gamma))$ converges for all idéles $z \in \mathbb{I}_K$ and for every adéle $y \in \mathbb{A}_K$, uniformly for y.

(iii) $\sum_{\gamma \in K} \left|\hat{f}(z\gamma)\right|$ is convergent for every idéle $z \in \mathbb{I}_K$.

Then,

$$\tilde{f} = \tilde{\hat{f}}$$

where, $\hat{f}$ denotes the Adélic Fourier Transform of f, and, $\tilde{f}(x) := \sum_{\gamma \in K} f(\gamma + x)$, $\forall x \in \mathbb{A}_K$. In other words,

$$\sum_{\gamma \in K} f(\gamma + x) = \sum_{\gamma \in K} \hat{f}(\gamma + x) \qquad ...(3.1)$$

Proof: We consider any function φ on $\mathbb{A}_K/K$, induced by some K-invariant function φ on $\mathbb{A}_K$.

Therefore, by definition (1.6.1) of Fourier Transform, we have,

$$\hat{\varphi}(z) = \int_{\mathbb{A}_K/K} \varphi(t)\psi(tz)\overline{dt}, \ \forall\ z \in K$$

where, ψ is a non-trivial continuous unitary character on $\mathbb{A}_K$; and, $\overline{dt}$ is the quotient measure on $\mathbb{A}_K/K$ induced by the measure dt on $\mathbb{A}_K$; satisfying the relation:

$$\int_{\mathbb{A}_K/K} \hat{f}(t)\overline{dt} = \int_{\mathbb{A}_K/K} \sum_{\nu \in K} f(\nu + t)\overline{dt} = \int_{\mathbb{A}_K} f(t)\,dt$$

for every $f \in S(\mathbb{A}_K)$ such that, f satisfies certain convergence properties.

It is important to mention that, each of the integrals in the above identity is well-defined, since, in the first two integrals, the integration variable t assumes value from the quotient group $\mathbb{A}_K/K$.

In order to prove the theorem, we shall need to apply the following two important lemmas:

Lemma 3.2.

For every continuous function $f \in S(\mathbb{A}_K)$,

$$\hat{f}\Big|_K = \hat{\tilde{f}}\Big|_K$$

Proof: Using definition of Fourier Transform, we obtain,

$$\hat{\tilde{f}}(z) = \int_{\mathbb{A}_K/K} \tilde{f}(t)\psi(tz)\overline{dt} = \int_{\mathbb{A}_K/K} \left(\sum_{\gamma \in K} f(\gamma + t)\right)\psi(tz)\overline{dt}, \left(Expanding\ \tilde{f}\right)\forall z \in K$$

$$= \int_{\mathbb{A}_K/K} \left(\sum_{\gamma \in K} f(\gamma + t)\psi\left((\gamma + t)z\right)\right)\overline{dt}$$

[Since ψ being assumed to be unitary on $\mathbb{A}_K$, hence, $\psi\,|_K = 1 \Rightarrow \psi(tz) = \psi\left((\gamma + t)z\right)$, $\forall \gamma \in K$, hence the equality holds by the definition of quotient measure on $\mathbb{A}_K$ relative to the counting measure on K].

$$= \int_{\mathbb{A}_K} f(t)\psi(tz)\,dt$$

$$= \hat{f}(z). \qquad \forall z \in K$$

Lemma 3.3.

For any $f \in S(\mathbb{A}_K)$ and for every $z \in K$,

$$\tilde{f}(z) = \sum_{\gamma \in Z} \hat{\tilde{f}}(\gamma)\overline{\psi}(\gamma z)$$

where $\overline{\psi}$ denotes the complex conjugate of ψ.

Proof: We have, a priori by Lemma (3.2),

$\hat{f}\,|_K = \hat{\hat{f}}\,|_K$, for $f \in S(\mathbb{A}_K)$

Therefore, by the fact that, the sum, $\sum_{\gamma\in K} \hat{f}(\gamma)\bar{\psi}(\gamma z)$ is normally convergent, we can assert that, the sum, $\sum_{\gamma\in K} \hat{\hat{f}}(\gamma)\bar{\psi}(\gamma z)$ is also normally convergent, i.e., precisely,

$\sum_{\gamma\in K} \left|\hat{\hat{f}}(\gamma)\right| < \infty$ [Since, ψ is unitary, hence, $\bar{\psi}$ is also unitary.]

Since we have the counting measure on K, also also due to the fact that, the Pontryagin Dual of $\mathbb{A}_K/K$ is K itself under the discrete topology, hence using Fourier Inversion Formula, we obtain,

$$\tilde{f}(z) = \sum_{\gamma\in K} \hat{\hat{f}}(\gamma)\bar{\psi}(\gamma z),\ \forall\, z \in K.$$

And our claim is established.

Putting $z = 0$ in Lemma (3.3), we have,

$$\tilde{f}(0) = \sum_{\gamma\in K} \hat{\hat{f}}(\gamma) = \sum_{\gamma\in K} \hat{f}(\gamma)$$

Although,

$\tilde{f}(0) = \sum_{\gamma\in K} f(\gamma)$ [Using definition of $\tilde{f}$]

And, hence,

$$\sum_{\gamma\in K} f(\gamma) = \sum_{\gamma\in K} \hat{f}(\gamma)$$

$\Rightarrow \tilde{f} = \tilde{\hat{f}}$. [Since, f is K-invariant on $\mathbb{A}_K$]

And the Poisson Summation formula is established.

Remark 3.4.

The sum, $\sum_{\gamma\in K} f(\gamma x)$ for every $x \in \mathbb{A}_K$ is defined as the Average for an Idéle x in $\mathbb{A}_K$.

Next, we shall introduce the statement of the Riemann-Roch Theorem which goes as follows:

Theorem 3.5: Riemann-Roch Theorem

Suppose $f \in S(\mathbb{A}_K)$, i.e., f satisfies the following conditions:

(i) $f \in L^1(\mathbb{A}_K)$, and, f is continuous.

(ii) $\sum_{\gamma\in K} f(x+\gamma)$ converges for all adéles $x \in \mathbb{A}_K$, uniformly.

(iii) $\sum_{\gamma\in K} \left|\hat{f}(\gamma)\right|$ is convergent.

Then,

$$\sum_{\gamma\in K} f(\gamma x) = \frac{1}{|x|}\sum_{\gamma\in K} \hat{f}(\gamma x^{-1}) \qquad ...(3.2)$$

Proof: Fix, $x \in \mathbb{I}_K$. Now, we define $h \in S(\mathbb{A}_K)$ by,

$$h(y) := f(yx), \ \forall y \in \mathbb{A}_K.$$

Then,

$$\sum_{\gamma \in K} h(\gamma) = \sum_{\gamma \in K} \hat{h}(\gamma) \text{ [ApplyingPoisson Summation Formula]} \quad ...(3.3)$$

Although, by definition of Fourier Transform,

$$\hat{h}(\gamma) = \int_{\mathbb{A}_K} f(tx)\psi(t\gamma)\,dt$$

$$= \frac{1}{|x|}\int_{\mathbb{A}_K} f(w)\psi(\gamma w x^{-1}), \text{ [Substituting, } w = tx \text{ in the integral]}$$

$$= \frac{1}{|x|}\hat{f}(\gamma x^{-1}) \text{ [By Definition (1.6.1)]}$$

Applying Poisson Summation Formula (Theorem (3.1), and from (3.3), we get,

$$\sum_{\gamma \in K} f(\gamma x) = \frac{1}{|x|}\sum_{\gamma \in K} \hat{f}(\gamma x^{-1})$$

And our theorem is established.

4. The Main Theorem

Theorem 4.1.

The following holds true for the Global Zeta Function $\zeta(f, \chi)$ of a number field K:

1. $\zeta(f, \chi)$ has a meromorphic extension on $\mathbb{C}$.
2. The extended global zeta function $\zeta(f, \chi)$ is holomorphic everywhere except when, $\mu = |.|^{-iy}$, $y \in \mathbb{R}$; hence having simple poles at the points $s = iy$ and $s = 1 + iy$ with residues given by,

   $-kf(0)$, and, $k\hat{f}(0)$

   respectively. Here, we can deduce that, $k := Vol.(C_K^1)$ = Volume of the Idéle Class Group of K of norm 1.
3. $\zeta(f, \chi)$ satisfies the functional equation,

$$\zeta(f, \chi) = \zeta(\hat{f}, \breve{\chi}) \quad ...(4.1)$$

where $\hat{f}$ denotes the Fourier Transform of a function $f \in S(\mathbb{A}_K)$ and, $\breve{\chi} := \chi^{-1}|.|$ is termed as the Shifted Dual of the character χ.

Proof: 1. For a number field K, a priori, we may write,

$$\zeta(f, \chi) = \int_0^\infty \zeta_t(f, \chi)\frac{1}{t}dt \quad ...(4.2)$$

for every quasi-character χ of exponent greater than 1. Where, we define,

$$\zeta_t(f, \chi) := \int_{\mathbb{I}_K} f(tx)\chi(tx)\,d^*x \quad ...(4.3)$$

Important to mention that, for any $x := (x_\nu)_\nu \in \mathbb{A}_K$, $t \in \mathbb{R}$, we define the element,

$$tx := (x'_\nu);\ \text{where, } x'_\nu := \begin{cases} x_\nu, & \text{if } \nu \neq \nu' \\ tx_\nu, & \text{if } \nu = \nu' \end{cases}.$$

For some specific non-Archimedean place ν'.

Our aim is to first establish the Functional Equation for $\zeta_t(f, \chi)$ using the Riemann-Roch Theorem proved earlier, which leads us to prove the following proposition:

Proposition 4.2.

The function $\zeta_t(f, \chi)$ satisfies the functional equation,

$$\zeta_t(f, \chi) = \zeta_{t^{-1}}(\hat{f}, \breve{\chi}) + \hat{f}(0) \int_{C_K^1} \breve{\chi}\left(\frac{x}{t}\right) d^*x - f(0) \int_{C_K^1} \chi(tx)\, d^*x \qquad \text{...(4.4)}$$

Proof: Using the Definition (1.5.9) of C_K^1,

$$\zeta_t(f, \chi) = \int_{1_K^1} f(tx)\chi(tx)\, d^*x = \int_{C_K^1} \left(\sum_{a\in K^*} f(atx)\right)\chi(tx)\, d^*x = \int_{C_K^1} \chi(tx)\, d^*x \left(\sum_{a\in K^*} f(atx)\right)$$

[Since $\chi|_{K^*} = 1$, by the hypothesis]

therefore, we get,

$$\zeta_t(f, \chi) + f(0) \int_{C_K^1} \chi(tx)\, d^*x = \int_{C_K^1} \chi(tx)\, d^*x \left(\sum_{a\in K^*} f(atx)\right)$$

And, now, we apply the Riemann-Roch Theorem, mentioned a priori, on the summand above so that, the Right Hand Side above yields the expression,

$$\int_{C_K^1} \chi(tx)\, d^*x \left(\sum_{a\in K} f(atx)\right) = \int_{C_K^1} \chi(tx)\, d^*x \left(\frac{1}{|tx|}\sum_{a\in K} f(atx)\right)$$

$$= \int_{C_K^1} \frac{\chi(tx)}{|tx|} d^*x \left(\sum_{a\in K} \hat{f}\left(at^{-1}x^{-1}\right)\right)$$

$$= \int_{C_K^1} \left|t^{-1}x\right| \chi\left(tx^{-1}\right) d^*x \left(\sum_{a\in K} \hat{f}\left(at^{-1}x\right)\right) \text{[Substituting } x \text{ by } x^{-1}\text{]}$$

$$= \zeta_{t^{-1}}(\hat{f}, \breve{\chi}) + \hat{f}(0) \int_{C_K^1} \breve{\chi}\left(\frac{x}{t}\right) d^*x$$

[A priori, from Equation (4.3), substituting t, f, χ by, t^{-1}, $\hat{f}$, $\breve{\chi}$ respectively].

$$\zeta_t(f, \chi) = \zeta_{t^{-1}}(\hat{f}, \breve{\chi}) + \hat{f}(0) \int_{C_K^1} \breve{\chi}\left(\frac{x}{t}\right) d^*x - f(0) \int_{C_K^1} \chi(tx)\, d^*x$$

And the result is established.

Using the above proposition, we shall prove our main theorem.

Proof: Using Equation (4.2), as obtained from the definition of $\zeta(f, \chi)$, and applying properties of integration, we obtain that,

$$\zeta(f,\chi)=\int_0^1 \zeta_t(f,\chi)\frac{1}{t}dt+\int_1^\infty \zeta_t(f,\chi)\frac{1}{t}dt$$

$$=: I_1+I_2 \text{ (say)} \qquad ...(4.5)$$

where,

$$I_1 := \int_0^1 \zeta_t(f,\chi)\frac{1}{t}dt \qquad ...(4.6)$$

And,

$$I_2 := \int_1^\infty \zeta_t(f,\chi)\frac{1}{t}dt \qquad ...(4.7)$$

Now,

$$I_2 := \int_1^\infty \zeta_t(f,\chi)\frac{1}{t}dt = \int_{\{x\in \mathbb{I}_K \| x| \geq 1\}} f(x)\chi(x)d^*x \text{ [Using Definition of } \zeta_t(f,\chi)]$$

Which is normally convergent for all $s \in \mathbb{C}$. [Since, the integral above on the R.H.S. is convergent for $\sigma = Re(s) > 1$]

Therefore, the integral I_2 is convergent for all $s \in \mathbb{C}$.

Using the functional equation deduced in the Proposition (4.2), for $\zeta_t(f,\chi)$, we obtain,

$$I_1 := \int_0^1 \zeta_t(f,\chi)\frac{1}{t}dt = \int_0^1 \zeta_{t^{-1}}\left(\hat{f},\breve{\chi}\right)\frac{1}{t}dt+\varepsilon \qquad ...(4.8)$$

Where the error term ε is defined as,

$$\varepsilon := \int_0^1 \left\{ \hat{f}(0)\int_{C_K^1} \breve{\chi}\left(\frac{x}{t}\right)d^*x - f(0)\int_{C_K^1} \chi(tx)d^*x \right\}\frac{1}{t}dt \qquad ...(4.9)$$

$$= \int_0^1 \left\{ \hat{f}(0)\breve{\chi}\left(t^{-1}\right)\int_{C_K^1} \breve{\chi}(x)d^*x - f(0)\chi(t)\int_{C_K^1} \chi(x)d^*x \right\}\frac{1}{t}dt \qquad ...(4.10)$$

Usng the fact that,

$$\int_0^1 \zeta_{t^{-1}}\left(\hat{f},\breve{\chi}\right)\frac{1}{t}dt = \int_1^\infty \zeta_t\left(\hat{f},\breve{\chi}\right)\frac{1}{t}dt \text{ [Sustituting } t^{-1} \text{ for } t] \qquad ...(4.11)$$

Using the result mentioned in Equation (1), we can say that, the integral, $\int_1^\infty \zeta_t\left(\hat{f},\breve{\chi}\right)\frac{1}{t}dt$ is normally convergent for all $s \in \mathbb{C}$, we assert that, the integral, $\int_0^1 \zeta_{t^{-1}}\left(\hat{f},\breve{\chi}\right)\frac{1}{t}dt$ also converges normally for every $s \in \mathbb{C}$.

Hence, we need to verify the convergence of only the error term ε in order to conclude that, the Global Zeta Function $\zeta(f,\chi)$ is normally convergent for every $s \in \mathbb{C}$.

Now, by definition of χ and $\breve{\chi}$, they are orthogonal, by orthogonality relations, we assert that,

$$\int_{C_K^1} \chi(x)d^*x = 0, \text{ and, } \int_{C_K^1} \breve{\chi}(x)d^*x = 0 \text{ [Since, } \chi \text{ is non-trivial on } \mathbb{I}_K^1]$$

Therefore, from Equation (4.9), we obtain, $\varepsilon = 0$ for χ non-trivial on $\mathbb{I}_K^1$.

If, χ is trivial on $\mathbb{I}_K^1$, a priori, we have the representation,

$\chi = \mu|.|^s$

Then, we can write,

$\chi = |\cdot|^{s'}$, where, $s' = s - i\tau$, for some $\tau \in \mathbb{R}$ [Since μ is unitary]

hence, evaluating ε using above expression for χ, we get,

$$\varepsilon = \int_0^1 \left\{ \hat{f}(0) t^{s'-1} Vol\left(C_K^1\right) - f(0) t^{s'} Vol\left(C_K^1\right) \right\} \frac{1}{t} dt$$

$$= Vol\left(C_K^1\right) \left\{ \frac{\hat{f}(0)}{s'-1} - \frac{f(0)}{s'} \right\} \left[\text{Since, } \chi|_{\mathbb{I}_K^1} = 1 \Rightarrow \breve{\chi}|_{\mathbb{I}_K^1} = 1 \Rightarrow \breve{\chi}|_{C_K^1} = 1\right]$$

Therefore, using Equation (1) and (4.8), we conclude that, $\zeta(f, \chi)$ is normally convergent for every s, and since, ε is a meromorphic function, thus we obtain our desired meromorphic extension of $\zeta(f, \chi)$ over $\mathbb{C}$

From above, we have, when χ is non-trivial on $\mathbb{I}_K^1$, then, $\mu \neq |\cdot|^{-i\tau}$, $\tau \in \mathbb{R}$, then, $\varepsilon = 0$. Hence, $\zeta(f, \chi)$ is Holomorphic everywhere.

When, χ is trivial on $\mathbb{I}_K^1$, then, $\mu = |.|^{-i\tau}$, $\tau \in \mathbb{R}$. Hence,

$$\varepsilon = Vol\left(C_K^1\right) \left\{ \frac{\hat{f}(0)}{s'-1} - \frac{f(0)}{s'} \right\} \text{[From (1)]}$$

Hence $\zeta(f, \chi)$ is holomorphic everywhere, except at the points, $s = i\tau$ and, $s = 1 + i\tau$, for $\tau \in \mathbb{R}$.

The respective residues can be evaluated as:

$-Vol\left(C_K^1\right) f(0)$, and, $Vol\left(C_K^1\right) \hat{f}(0)$

i.e., $-kf(0)$, and, $k\hat{f}(0)$

respectively, where, we have, $k := Vol.\left(C_K^1\right)$

Using the identities in (1) and (4.8), we get, for the Global Zeta Functions $\zeta(f, \chi)$,

$$\zeta(f, \chi) = \int_0^1 \zeta_t(f, \chi) \frac{1}{t} dt + \int_0^1 \zeta_{t^{-1}}\left(\hat{f}, \breve{\chi}\right) \frac{1}{t} dt + \varepsilon(f, \chi)$$

$$= \int_0^1 \left(\int_{\mathbb{I}_K} f(tx) \chi(tx) d^* x \right) \frac{1}{t} dt + \int_0^1 \left(\int_{\mathbb{I}_K} \hat{f}(tx) \breve{\chi}(tx) d^* x \right) \frac{1}{t} dt + \varepsilon(f, \chi) \qquad \text{...(4.12)}$$

Moreover, applying propperties of Fourier Transform for $f \in S(\mathbb{A}_K)$, we get,

$$\hat{\hat{f}} = f(-x), \text{ and, } \breve{\breve{\chi}} = \chi \qquad \text{...(4.13)}$$

Substituting $\hat{f}$ and $\breve{\chi}$ instead of f and χ in Equation (4.12), we get,

$$\zeta\left(\hat{f}, \check{\chi}\right)=\int_1^\infty \zeta_t\left(\hat{f}, \check{\chi}\right)\frac{1}{t}dt+\int_1^\infty \zeta_{t-1}\left(\hat{\hat{f}}, \chi\right)\frac{1}{t}dt+\varepsilon\left(\hat{f}, \check{\chi}\right)$$

$$=\int_1^\infty\left(\int_{\mathrm{I}_K}\hat{f}(tx)\check{\chi}(tx)d^*x\right)\frac{1}{t}dt+\int_1^\infty\left(\int_{\mathrm{I}_K}f(-tx)\chi(tx)d^*x\right)\frac{1}{t}dt+\varepsilon\left(\hat{f}, \check{\chi}\right) \quad \text{...(4.14)}$$

But, from (4.9), we have,

$$\varepsilon(f, \chi)=\int_0^1\left\{\hat{f}(0)\check{\chi}\left(t^{-1}\right)\int_{C_K^1}\check{\chi}(x)d^*x-f(0)\chi(t)\int_{C_K^1}\chi(x)d^*x\right\}\frac{1}{t}dt$$

Hence,

$$\varepsilon\left(\hat{f}, \check{\chi}\right)=\int_0^1\left\{\hat{\hat{f}}(0)\check{\check{\chi}}\left(t^{-1}\right)\int_{C_K^1}\check{\check{\chi}}(x)d^*x-\hat{f}(0)\check{\chi}(t)\int_{C_K^1}\check{\chi}(x)d^*x\right\}\frac{1}{t}dt$$

$$=\int_0^1\left\{\hat{f}(0)\check{\chi}\left(t^{-1}\right)\int_{C_K^1}\check{\chi}(x)d^*x-f(0)\chi(t)\int_{C_K^1}\chi(x)d^*x\right\}\frac{1}{t}dt$$

[Using properties (4.13)]

$= \varepsilon(f, \chi)$

Showing that, ε is invariant under the transformation map,

$$(f\ \chi)\mapsto\left(\hat{f}, \check{x}\right)$$

Also, given the fact that, $\chi = \mu|.|^s$, and also χ being invariant under the tranformation map,

$tx \mapsto -tx$

Therefore, $\chi(tx) = \chi(-tx)$ for every $t \in \mathbb{R}$ and, $x \in \mathrm{I}_K$, since χ is an Id\`ele Class Character.

Therefore, substituting $\chi(-tx)$ instead of $\chi(tx)$ in the second integral in the Equation (4.14), we obtain,

$$\zeta(f, \chi)=\zeta\left(\hat{f}, \check{\chi}\right)$$

Which establishes our desired result and completes the proof of the theorem.

Remark 4.3.

In the Theorem (4.1) the volume of the Id‘ele Class Group of norm 1, $C_K^1 := \mathrm{I}_K^1 / K^*$ is measured with respect to the Haar Measure on C_K defined by d^*x and the counting measure on K^*. □

Remark 4.4. □

By further calculation it can be deduced that,

$$Vol.\left(C_K^1\right)=Res_{s=1}\zeta_K(s)$$

where, $\zeta_K(s)$ denotes the Dedekind Zeta Fumction on the global field K.

Remark 4.5.

If we consider the global field K to be a funcion Field instead of a number field, the same statement of the main theorem mentioned above holds true for the Global Zeta Functions $\zeta(f, \xi)$ of K, although the proof differs significantly from that in case of the number fields.

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