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The Branching Theorem and Semi-Cofinite Topologies

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Abstract

It is proved that every nonempty set X has a chain of topologies with the cofinite topology as its finest (maximum). For their semblance to, and yet differences from the cofinite topology, these other topologies in the chain are called semi-cofinite topologies. We proved that some of the semi-cofinite topologies in the chain are themselves the maxima of yet other sequences of pair-wise comparable semi-cofinite topologies on the nonempty set X. The cofinite topology lemma and the cofinite topology theorem are stated and proved. The entire exposition climaxed into what we finally called the Branching Theorem. The interesting meaning of the branching theorem is that every nonempty set is-topologically speaking—a tree of many branches and sub-branches of topologies that are pair-wise comparable.

Keywords: *Topology, Finer, Coarser, Weaker and stronger topologies, Comparable topologies*

1. Introduction

Definition 1.1.: *Let X be an infinite set and let*

$C = \{A \subset X : A^c \text{ is finite}\} \cup \{\phi\}$. *Then C is a topology on X, called* the *cofinite topology on X.*

Remark: Some authors have defined *cofinite* topology in other ways. For example, in example 3 on page 77 of (James, 2000) we observe the following: "Let X be a set; let T_f be the collection of all subsets U of X such that X-U either is finite or is all of X. Then T_f is a topology on X, called the finite complement topology." One can also see (Seymour, 1965), page 66, for a seemingly different but same definition of cofinite topology. It has to be said here, though, that the meaning in all these definitions are the same.

Some of the well-known properties of the cofinite topology C on a set X are as follows:

1. For an infinite set X, the complement of every C-open set (apart from the empty set) is finite—this is the actual *complement finite* or *co-finite*ness property.

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2. If X is infinite, then C has infinitely many open sets.
3. If X is infinite, then C is not closed under arbitrary intersections.
4. There is one and only one cofinite topology C on a set X.
5. The cofinite topology C on a set X is always T_1; indeed the cofinite topology is the strongest T_f-topology on any set X.

The properties of a cofinite topology outlined above will soon be compared and contrasted with those of *semi-cofinite* topologies defined and constructed below in the next section.

2. Main Results—Peak of a Sequence of Pair-Wise Comparable Topologies

Definition 2.1.: *If a collection C_s of subsets of an infinite set X is such that C_s contains the empty set and that every (other) set in C_s has finite complement, then C_s is called a semi-cofinite topology on X if it is a strictly weaker topology than the cofinite topology on X.*

The following are the properties against which each semi-cofinite topology, when constructed, will be checked.

1. The complement of every C_s-open set (apart from the empty set) is finite.
2. If X is infinite, then C_s can have infinitely many open sets or only a finite number of open sets—depending on how we choose to construct C_s.
3. If X is infinite, C_s can be closed under arbitrary intersections, or not closed under arbitrary intersections—depending on how C_s is constructed.
4. A set X can (and do often) have more than one semi-cofinite topology.
5. No semi-cofinite topology C_s on a set X is T_1.

So, one main difference between a cofinite topology C and a semi-cofinite topology C_s is that C is always T_1 and C_s is never T_1. And one special relationship between a cofinite topology C and a semi-cofinite topology C_s is that C_s is always strictly weaker than C, on X. Last, but not the least, the cofinite topology C and each semi-cofinite topology C_s have the *co-finite or complement finite* property in common. These differences and similarities necessitated the new definition—for if only one topology has a name, then a very large class of other topologies related to the named topology should have their name in autonomy.

Example 1:

Let $N = \{0, 1, 2, ...\}$ denote the set of natural numbers. For each $n \in N$ let G_n be the set of all real numbers *excluding* the first n natural numbers. Thus for instance

$G_0 = R - \{\} = R$;

$G_1 = R - \{0\}$;

$G_2 = R - \{0, 1\}$; $G_3 = R - \{0, 1, 2\}$;

...

$G_n = R - \{0, 1, 2, 3, ..., n-1\}$

Let $T_{CN} = \{\phi, G_n\}_{n \in N}$

Then it is easy to see that

1. The empty set is in T_{CN}, from the way T_{CN} is defined.
2. The whole set R of real numbers is in T_{CN}.
3. The complement of every set in T_{CN}, apart from the empty set, is finite; precisely contains the first n natural numbers.
4. And that T_{CN} is closed under finite intersections and arbitrary unions.
5. Hence T_{CN} is a topology on R, satisfying all but one property of the cofinite topology, on R, namely that it is not the

family of *all* subsets of R whose complements are finite, together with the empty set. Hence T_{CN} is an example of what we, in this article, call *semi-cofinite* topology, on the set R of real numbers. It is strictly weaker than the cofinite topology on R.

We shall call T_{CN} the semi-cofinite topology on R generated by the set N of natural numbers. (Observe that the set of natural numbers N can generate a semi-cofinite topology on R in a different way.)

Example 2:

Let $Z = \{0, 1, -1, 2, -2, 3, -3, ...\}$ denote the set of all integers, arranged thus. For each $n \in N = \{0, 1, 2, ...\}$, let G_n be R without the first n integers under the arrangement thus made of Z. Hence for instance

$G_0 = R$;

$G_1 = R - \{0\}$;

$G_2 = R - \{0, 1\}$;

$G_3 = R - \{0, 1, -1\}$;

$G_4 = R - \{0, 1, -1, 2\}$;

$G_5 = R - \{0, 1, -1, 2, -2\}$;

...

etc.

Then $T_{CZ} = \{\phi, G_n\}_{n\in N}$ is a semi-cofinite topology (different from the one above) on R, as can easily be verified.

We observe that in T_{CN}, $G_2 = R - \{0, 1\}$ and that in T_{CZ}, $G_2 = R - \{0, 1\}$ also. However in T_{CN}, $G_3 = R - \{0, 1, 2\}$ and in T_{CZ}, $G_3 = R - \{0, 1, -1\}$ and we see that though G_2 is common to both T_{CN} and T_{CZ}, G_3 is not common to the two topologies on R. It may further be verified that G_n is not common to the two topologies if $n > 3$. Hence these two topologies are not comparable. But we can obtain two comparable semi-cofinite topologies on R based on the two subsets N and Z. Let Z be written in the alternative (and usual) form $Z = \{..., -2, -1, 0, 1, 2, ...\}$ and let $G_n = R - \{n \text{ integers}\}$ for each $n \in N = \{0, 1, 2, ...\}$. That is, G_n is R without a finite number of whole numbers. We can also observe that the complement of each G_n is a finite number of integers. This observation is helpful in proving that the family $T_Z = \{\phi, G_n\}_{n\in N}$ is closed under arbitrary unions; for if $\{G_\alpha\}_{\alpha\in\Delta} \subset T_Z$ is any family of sets in T_Z, then $\bigcap_{\alpha\in\Delta} G_\alpha^c$ is a finite number of integers, being an intersection of finite sets of integers. It follows that:

$$\left(\bigcap_{\alpha\in\Delta} G_\alpha^c\right)^c = \bigcup_{\alpha\in\Delta} G_\alpha$$

is the set R of real numbers without n integers. Similarly we see that

$$\bigcap_{i=1}^{n} G_i = \left(\bigcup_{i=1}^{n} G_i^c\right)^c$$

is the complement of a finite number of whole numbers in R. Hence T_Z is a semi-cofinite topology on R. And it is easy to see that T_{CN} is strictly weaker than T_Z. There are other ways of constructing strictly comparable pairs of semi-cofinite topology on any infinite set. Later developments here will show that.

Example 3:

Let $P = \{p_1, p_2, p_3, ...\}$ be the ordered (ascendingly) set of all prime numbers, and R and N as earlier introduced. Let G_n denote R without the first n prime numbers. Then $T_{CP} = \{\phi, G_n\}_{n\in N}$ is yet another semi-cofinite topology on R, different from the two introduced earlier.

Example 4:

Let $Q = \{q_1, q_2, q_3, ...\}$ be the set of all rational numbers, and R and N as earlier introduced. Put $G_n = R - \{n \text{ rational numbers}\}$. Then $T_{CQ} = \{\phi, G_n\}_{n\in N}$ is another semi-cofinite topology on R.

Example 5:

Let Q^c be the set of all irrational numbers, and R and N remain as introduced before. Put $G_n = R - \{n \text{ irrational numbers}\}$, and let $T_{CQ}{}^c = \{\phi, G_n\}_{n\in N}$. Then = is a semi-cofinite topology on R.

Example 6:

With R and N as earlier introduced, let $G_n = R - \{n \text{ real numbers }\}$. Put $T_{CR} = \{\phi, G_n\}_{n\in N}$. Then T_{CR} is *the* cofinite topology on R.

Note: We observe that the topology constructed last above (i.e., example 6), T_{CR}, is the family of all subsets of R whose complements are finite, together with the empty set—hence only this topology is the cofinite topology on R. The general method of constructing cofinite topologies supplied here can be used to construct cofinite topology on any set. For example if we put $N = R$ then example 1 will coincide with what is found in Seymour (1965). If $N = R$ in example 6 we get the cofinite topology on N.

We have proved in the remarks following example 1 that all the constructions (1 to 6) are semi-cofinite topologies. It has also been proved by other authors that the family constructed in definition 1 is indeed a topology on X. We provide below an alternative, rigorous and particular proof that the family in example 6 is indeed the cofinite topology on the set R of real numbers.

Proposition 2.1.: *The family T_{CR} as constructed in example 6 is* the *cofinite topology on the set R of real numbers.*

Proof: We only prove that T_{CR} is closed under finite intersections and arbitrary unions, since the remaining two properties of a (cofinite) topology are easily seen to be satisfied by T_{CR}; we also note that T_{CR} is a T_1-space (property of all cofinite topologies) as singletons are T_{CR}-closed subsets of R. We recall that

$G_0 = R$;

$G_1 = R - \{r_{11}\}$;

$G_2 = R - \{r_{21}, r_{22}\}$;

$G_3 = R - \{r_{31}, r_{32}, r_{33}\}$;

$G_4 = R - \{r_{41}, r_{42}, r_{43}, r_{44}\}$;

$G_n = R - \{r_{n1}, r_{n2}, r_{n3}, ..., r_{nn}\}$; where $r_{ni} \in R$.

Now let $N = \bigcap_{k=1}^{n} G_k = \bigcap_{k=1}^{n}\left(R - \{r_{k1}, r_{k2}, r_{k3}, \cdots, r_{kk}\}\right)$

$= R - \bigcup_{k=1}^{n}\{r_{k1}, r_{k2}, r_{k3}, \cdots, r_{kk}\}$

Hence the complement N_c of N is $N^c = \bigcup_{k=1}^{n}\{r_{k1}, r_{k2}, r_{k3}, \cdots, r_{kk}\}$, a finite union of finite sets; and so it must be finite.

Alternatively we can consider directly the cardinality of N^c. Card (N^c) is such that Card $\left(N^c\right) \le 1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2} < \infty$. (The numbers 1, 2, 3, ..., n added represent the cardinalities of the complements of the G_k, for $k = 1, 2, ..., n$) So N^c is finite, implying that T_{CR} is closed under finite intersections.

Now let $U = \bigcup_{\alpha\in\Delta} G_\alpha \equiv \bigcup_{m\ge t} G_m = \bigcup_{m\ge t}\left(R - \{r_m\},\ i = 1,\cdots,m\right) = R - \left(\bigcap_{m\ge t}\{r_{mi}\},\ i = 1,\cdots,m\right)$

Hence the complement of U is $U^c = \left(\bigcap_{m\ge t}\{r_{mi}\},\ i = 1,\cdots,m\right) \subseteq \{r_{ti}\},\ i = 1,\cdots t$. Hence Card ($U^c$) < $t < \infty$, implying that T_{CR} is also closed under arbitrary unions.

2.1. Observations

We have said that the topologies T_{CN} and T_{CZ} (in examples 1 and 2) are not comparable, from the observation that G_3 in

T_{CN} is $G_3 = R - \{0, 1, 2\}$ while in T_{CZ}, $G_3 = R - \{0, 1, -1\}$. Hence G_3 in T_{CN} is not the same as G_3 in T_{CZ}. If we denote G_n in T_{CN} by $T_{CN}(G_n)$, and G_n in T_{CZ} by $T_{CZ}(G_n)$ then it is easy to see that $T_{CN}(G_n) \notin T_{CZ}$ if $n \in N$ and $n > 3$; and conversely, $T_{CZ}(G_n) \notin T_{CN}$ if $n \in N$ and $n > 3$, though $T_{CN}(G_n) = T_{CZ}(G_n)$ if $n = 0, 1, 2$.

However, if we define G_n in T_{CN} by $G_n = R - \{n \text{ natural numbers}\}$, and define G_n in T_{CZ} by $G_n = R - \{n \text{ whole numbers}\}$, then it would be seen that, since the set N of natural numbers is a subset of the set Z of integers, the collection T_{CN} of the set of real numbers without some natural numbers is a sub-collection of the collection T_{CZ} of R without some integers. Hence the semi-cofinite topology T_{CN} on R (this time) is strictly weaker than the semi-cofinite topology T_{CZ} on R. That is, $T_{CN} < T_{CZ}$. Similarly $T_{CP} < T_{CZ}$, and $T_{CZ} < T_{CQ}$. And all the five semi-cofinite topologies, on R, above can be summarized as follows:

1. $T_{CN} < T_{CZ} < T_{CQ} < T_{CR}$;
2. $T_{CP} < T_{CZ} < T_{CQ} < T_{CR}$;
3. $T_{CQ}{}^{c} = < T_{CR}$; and
4. $T_{CP} < T_{CN}$.

The procedure for constructing the semi-cofinite topologies on R, above, can be applied to any infinite set and summaries similar to 1 to 4 can be put under a lemma as follows:

Lemma 2.1 (The Cofinite Topology Lemma): *Let A and B be two infinite subsets of an infinite set X such that $A \subset B$ (where A is a proper subset of B). Then there exist semi-cofinite topologies T_{CA} and T_{CB}, on X, induced respectively by A and B such that $T_{CA} < T_{CB}$; that is, T_{CA} is strictly weaker than T_{CB}.*

Example 7:

Let X be an infinite set and let $B = X - \{x_1\}$ and $A = B - \{x_2\} = X - \{x_1, x_2\}$. Then A is an infinite and proper subset of B, and (W.L.O.G) B is an infinite and proper subset of X. Let

$G_0 = X - \{\} = X$;

$G_1 = B = G_0 - \{x_1\} \equiv X - \{x_1\}$;

$G_2 = A = G_1 - \{x_2\} = B - \{x_2\} \equiv X - \{x_1, x_2\}$;

$G_3 = G_2 - \{x_3\} = A - \{x_3\} = G_1 - \{x_2, x_3\} = B - \{x_2, x_3\} \equiv X - \{x_1, x_2, x_3\}$;

$$\begin{aligned} G_4 &= G_3 - \{x4\} \\ &= G_2 - \{x_3, x_4\} \\ &= G_1 - \{x_2, x_3, x_4\} \\ &= G_0 - \{x_1, x_2, x_3, x_4\} \end{aligned}$$

And so on,

$G_n = G_0 - \{x_1, x_2, ..., x_n\} = X - \{x_1, x_2, ..., x_n\}$.

Let $T_{CA} = \{\phi, G_0, G_2, G_3, ..., G_n, ...\}$. Then T_{CA} is a semi-cofinite topology on X. Let $T_{CB} = \{\phi, G_n\}_{n \in N}$. Then T_{CB} is another semi-cofinite topology on X. And we see that T_{CA} is a strictly weaker topology than T_{CB} on X since $G_1 \in T_{CB}$ and $G_1 \notin T_{CA}$ and every T_{CA}-open set is T_{CB}-open.

We also see that the open sets of both semi-cofinite topologies satisfy the inclusions

$\cdots \subset G_3 \subset G_2 \subset G_0 = X$, for T_{CA};

and

$\cdots \subset G_3 \subset G_2 \subset G_1 \subset G_0 = X$, for T_{CB}.

Hence both semi-cofinite topologies are closed under arbitrary intersections, making them complement topologies.[1] Finally, we remark that one must not follow the process of construction used here to have two comparable semi-cofinite

[1] Complement topology is an idea originated by the authors in one of their publications in 2017. See (Chika and Alexander, 2017).

topologies induced on X by A and B. For example we could have used procedures exactly similar to those used in examples 4 and 5, and still have T_{CA} to be strictly weaker than T_{CB}—only that this time the semi-cofinite topologies may not be closed under arbitrary intersections.

Let C_N and C_Z be respectively the cofinite topology on N and Z. Let $T_N = C_N \cup \{Z\} \cup \{R\}$ and $T_Z = C_Z \cup \{R\}$. Then, by subset-induced topologies, both T_N and T_Z are topologies on R and T_N is strictly weaker than T_Z.

Finally, we observe that these two topologies are semi-cofinite topologies on R.

Remark: We know that every infinite set, say $X = \{x_1, x_2, x_3, ...\}$, has an infinite proper subset, say $X_1 = \{x_1, x_2, x_3, ...\}$. From the cofinite topology lemma, above, we can construct (or there exists) a semi-cofinite topology T_{CX_1} on X, and T_{CX}, such that $T_{CX} > T_{CX_1}$. Since X_1 is itself infinite, it has an infinite proper subset, say X_2. By the cofinite topology lemma again, there exists a semi-cofinite topology T_{CX_2} on X such that $T_{CX} > T_{CX_1} > T_{CX_2}$. The reasoning can continue like that, and what we have proved is the following:

Theorem 2.1. (Cofinite Topology Theorem): *Let X be any infinite set. There exists a sequence $\{\tau_1, \tau_2, \tau_3, ...\}$ of topologies on X, forming a chain in that*

$$C = T_{CX} > \tau_1 > \tau_2 > \tau_3 > \cdots$$

where $C = T_{CX}$ is the cofinite topology on X.

Further research may now be geared towards finding other topologies that may exist between or at the extreme ends of this chain.

3. Main Results-The Branching Theorem

Definition 3.1.: *A topology is called a chain-element topology if it is an element of a family of topologies that form a (decreasing or increasing) chain on a set.*

Note: One implication of the cofinite topology theorem is that every infinite set has infinitely many semi-cofinite topologies. The other implication is that an infinity of semi-cofinite topologies on any infinite set can be constructed to form a chain, on the top of which sits the cofinite topology of the set.

We observe that each of the semi-cofinite topologies so far constructed here has an infinite number of open sets. However, we also have to point out that they alone are not the only semi-cofinite topologies: there are some semi-cofinite topologies with only finitely many open sets. In fact, each of the chain element semi-cofinite topologies with infinite number of open sets can be shown to be (themselves) the limit of an increasing sequence of pair-wise comparable semi-cofinite topologies with finite numbers of open sets.

Example 8:

Let us take another look at example 1, the semi-cofinite topology T_{CN} on R generated by the set of natural numbers. If we serially collect finite numbers of open sets of T_{CN}, we shall get an increasing sequence of semi-cofinite topologies on R forming a chain at the top of which sits T_{CN}. To see this, we go as usual and let

$G_0 = R - \{\} = R;$

$G_1 = R - \{0\}.$

Then $\tau_1 = \{\phi, G_0, G_1\}$ is a semi-cofinite topology, on R, strictly weaker than T_{CN}.

Let $G_2 = R - \{0, 1\}$. Then, with G_0 and G_1 as earlier defined, $\tau_2 = \{\phi, G_0, G_1, G_2\}$ is yet another (semi-cofinite) topology, strictly weaker than T_{CN}, on R.

Continuing like that, with $G_n = R - \{0, 1, 2, ..., n-1\}$ and G_i $(1 < i < n-1)$ as earlier defined, we see that $\tau_n = \{\phi, G_k : k = 0, 1, ..., n\}$ is a (semi-cofinite) topology on R, strictly weaker than T_{CN}.

We finally observe that τ_1 is strictly weaker than τ_2, and τ_2 is strictly weaker than τ_3, and so on. That is

$\tau_1 < \tau_2 < \tau_3 < \cdots < T_{CN}$,

where T_{CN} is as earlier introduced in example 1. That is, some semi-cofinite topologies can branch out into the limit of another sequence of pair-wise comparable topologies. Since each of the chain element topologies (τ_n on X) in the cofinite topology theorem is induced on X by an infinite set, X_n, each of these semi-cofinite topologies can be made to sit at the top of yet another sequence of pair-wise comparable topologies. For example, if τ_n is induced on X by X_n, let X_{n_i} be X_n without i elements (where $i \in N$). That is, $X_{n_1} = X_n - \{x_1\}$, where $x_1 \in X_n$; $X_{n_2} = X_{n_1} - \{x_2\}$, where $x_2 \in X_{n_1}$; and so on. We see that $X_n \supset X_{n_1} \supset X_{n_2} \cdots$. It follows from the cofinite topology lemma that the semi-cofinite topology $T_{CX_{n_1}}$ on X is strictly weaker than $T_{CX_n} = \tau_n$; $T_{CX_{n_2}}$ is strictly weaker than $T_{CX_{n_1}}$; $T_{CX_{n_3}}$ is strictly weaker than $T_{CX_{n_2}}$; and so on.

That is

$$\ldots < T_{CX_{n_3}} < T_{CX_{n_2}} < T_{CX_{n_1}} < T_{CX_n} = \tau_n < T_{CX} = C, \cdots (*)$$

where C is the cofinite topology on X.

That is, the topology τ_n on X, induced by the subset X_n of X, sits at the top of a chain of pair-wise comparable topologies. Each X_{n_i}, subset of X_n, induces the topology $T_{X_{n_i}}$ on X, strictly weaker than τ_n, as seen in (*). By a process similar to what has been used to generate (*), under τ_n, we can have another chain

$$H = \left\{T_{X_{n_{ij}}}\right\}_{j=1}^{\infty}$$

of topologies on X, pair-wise comparable, and such that each $T_{X_{n_{ij}}}$ is strictly weaker than $T_{X_{n_i}}$. The process can continue for each of the (infinite) subsets X_{n_i} ($i = 1, 2, 3, \ldots$, and $n = 1, 2, 3, \ldots$) of X—and their own subsets. What we have proved is the following.

Theorem 3.1. (Branching): *Each of the chain element topologies under the cofinite topology theorem is itself at the peak of yet another chain of (semi-cofinite) topologies. If the original set X is infinite, then this branching will be endless; if X is finite, the branching will terminate.*

For example each of the semi-cofinite topologies 3 to 7 above, on R, is the limit of a sequence of pair-wise comparable monotone increasing semi-cofinite topologies.

Example 9:

We may again let $G_0 = N = \{0, 1, 2, \ldots\}$;

$G_1 = N - \{0\} = \{1, 2, 3, \ldots\}$.

Then $\tau_1 = \{\phi, G_0, G_1\}$ is a (semi-cofinite) topology on the set N of natural numbers. If we also let

$G_0 = N$, $G_1 = N - \{0\}$, $G_2 = N - \{0, 1\} = \{2, 3, 4, \ldots\}$,

then $\tau_2 = \{\phi, G_0, G_1, G_2\}$ is another topology, strictly stronger than τ_1, on N. If we continue like that, for each $n \in N$ then

$$\tau_n = \{\phi, G_k\}_{k=0}^{n}$$

is a semi-cofinite topology on N, strictly stronger than τ_{n-1}. We then see that the family $H_1 = \{\tau_n\}_{n=1}^{\infty}$ of topologies on N form an increasing chain of topologies, on N, at which peak lies the cofinite topology on N. We note that each chain element of H_1 has only finitely many open sets. This is to be contrasted with $H_2 = \{\tau_n\}_{n=1}^{\infty}$ whose elements $\tau_1 = T_{CX_1}$, $\tau_2 = T_{CX_2}$, $\tau_3 = T_{CX_3}$, $\ldots$, have each infinitely many open sets, where $X_1 = N - \{0\} = \{1, 2, 3, \ldots\}$, $X_2 = N - \{0, 1\}$

= {2, 3, 4, ...}, etc., and the topologies $\tau_1 = T_{CX_1}$, $\tau_2 = T_{CX_2}$, $\tau_3 = T_{CX_3}, \cdots$ are constructed according to the remark in Lemma 2.1.

3.1. Finite Sets

Since the complement of every subset of a finite set is finite, any search for subsets of a finite set whose complements are finite is not an interesting exercise. Hence we do not often talk about cofinite topologies on finite sets. However if X is finite, then 2^X the power set of X is the cofinite topology on X since it satisfies the definition of cofinite topology. Also if X is finite, say $X = \{x_1, x_2, x_3, ..., x_n\}$, then we have some semi-cofinite topologies on X, forming a chain also. These semi-cofinite topologies can be constructed as follows:

Let $G_0 = X$; $G_1 = X - \{x_1\}$. Then $\tau_1 = \{\phi, G_0, G_1\}$ is a semi-cofinite topology on X.

Let $G_2 = X - \{x_1, x_2\}$. Then $\tau_2 = \{\phi, G_k\!:\!k = 0, 1, 2\}$, where G_0, G_1 are as earlier defined, is another semi-cofinite topology on X, strictly stronger than τ_1. With $G_3 = X - \{x_1, x_2, x_3\}$, we see that $\tau_3 = \{\phi, G_k\}_{k=0}^{3}$ is another semi cofinite topology, strictly stronger than τ_2.

Continuing like that, with $G_n = X - \{x_1, x_2, x_3, ..., x_n\} = \phi$, we see that $\tau_n = \{\phi, G_k\}_{k=0}^{n}$ is a topology, strictly stronger than τ_{n-1}. That is

$$\tau = \tau_n = \{\phi, G_k\}_{k=0}^{n}$$

is a topology on X, stronger than all the other ones. So, we have a finite sequence $\{\tau_k\}_{k=1}^{n}$ of topologies on X forming a chain in that

$$\tau_n > \tau_{n-1} > \cdots > \tau_1,$$

and the power set 2^X of X, or its cofinite topology, is at the top of this finite sequence of topologies.

4. Conclusion and Summary

1. It is proved that every nonempty set X has a chain of topologies with the cofinite topology as its finest. We called these other topologies in the chain semi-cofinite topologies.
2. We proved that some of the semi-cofinite topologies in the chain are themselves the maxima of yet other sequences of pair-wise comparable semi-cofinite topologies on the nonempty set X.
3. The cofinite topology lemma and the cofinite topology theorem were stated and proved.
4. We stated and proved the Branching Theorem; and the implication of this theorem is that every nonempty set X is—topologically speaking—a tree of many branches and sub-branches of topologies that are pair-wise comparable.
5. For an infinite set, the branches and sub-branches of the tree of topologies can be endless; or be made to be finite.
6. Ample examples are given at appropriate places to clearly illustrate the theorems being developed.

References

Angus E. Taylor and David C. Lay. (1980). *An Introduction to Functional Analysis*, 2nd Edition, John Wiley and Sons, New York.

Chika Moore and Alexander Ilo. (2017). Complement Topologies. *Paper Published in the IOSR Journal of Mathematics (IOSR-JM),* 13(3), Ver. V (May-June), 73-77. www.iosrjournals.org ; Visible on Google Scholar

Edwards, R.E. (1995). Functional Analysis: Theory and Applications. Dover Publications Inc., New York.

James R. Munkres. (2000). *Topology*, 2nd Edition, Massachusetts Institute of Technology, Prentice Hall, Upper Saddle River, NJ 07458.

Royden, H.L. (2005). *Real Analysis*, 3rd Edition, Prentice-Hall of India Private Limited, New Delhi.

Rudin, W. (1973). *Functional Analysis*, McGraw-Hill, New York.

Seymour, Lipschutz. (1965). Theory and Problems of General Topology. *Schaum's Outline Series*, MacGraw-Hill Book Company.

Sheldon W. Davis. (2005). *Topology*, McGraw-Hill Higher Education, Boston.

Willard Stephen. (2004). *General Topology*, Courier Dover Publications.

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