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Proliferation of Compactness

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Abstract

Compactness is considered one of the most important properties to be possessed by a topological space X . Yet it is also believed that it is not easy to come across a compact space. So if a space X is not compact, mathematicians would look to see if there is any way through which what approximates to compactness can be achieved for X . This article breaks the invincibility of compactness; it establishes that any nonempty set X has as many topologies as (at least) the cardinality of X , each of which makes X compact. Furthermore, we showed that if $Card(X) = n$, then there exists a chain.

$$C = \{ \tau_\alpha : \alpha \in \Delta \}$$

of pairwise comparable topologies on X such that (X, τ_α) is a compact topological space, for each $\alpha \in \Delta$, and

$$Card(\Delta) = Card(C) \geq n$$

In short, we bumped into a Proliferation of Compact Topologies for any nonempty set X . In fact, if A is a subset of a nonempty set X , then there exist as many topologies as the cardinality of X namely $\{ \tau_x : x \in X \}$, on X , with respect to each of which the subset A is compact. This is a crucially strange result.

Keywords: *Topology, Compact topology, Finer, Coarser, Weaker and stronger topologies, Cofinite topology, Semi-cofinite topologies, Comparable topologies, Chain of topologies*

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1. Introduction

Definition 1.1: Let X be an infinite set and let $C = \{ A \subset X : A^c \text{ is finite} \} \cup \{ \emptyset \}$. Then C is a topology on X , called the cofinite topology on X .

Definition 1.2: If a collection C_s of subsets of an infinite set X is such that C_s contains the empty set and that every (other) set in C_s has finite complement, then C_s is called a semi-cofinite topology on X if it is a strictly weaker topology than the cofinite topology on X .

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Definition 1.3: Let (X, τ) be a topological space, and let A be a subset of X . If every open cover for A has a finite sub-cover for A , then A is called a compact set in the topological space.

Definition 1.4: Let (X, τ) be a topological space. If X is compact, then τ is called a compact topology on X .

Remark: This somewhat new definition (in 1.4) is not all that new; because the topologies of all the known compact spaces fit in to this definition—which has not really been in existence before now. However, it is easy to observe that if τ_1 and τ_2 are two topologies on a set X , then (X, τ_1) may be compact while (X, τ_2) is not compact. Because of this observation, definition 1.4 will be seen simply as a way of separating the topologies with respect to which X is compact from those with respect to which X is not compact. The definition also highlights the fact that compactness of a space X is a function of the topology around X (i.e., the *topological environment* of X) and not an intrinsic property or character of X as a set.

Example 1: The set R of real numbers in its usual topology is not compact, but as we shall soon see, R is compact under appropriate topologies.

Example 2: Let $X = R$ (the set of real numbers), $\tau_1 = \{X, \emptyset, \{2\}, \{3\}, \{2, 3\}\}$. Let τ_2 represent the usual topology of R . Then (X, τ_1) is compact while (X, τ_2) is not compact.

Lemma 1.1.: Any finite topology is a compact topology.

Proof: Let (X, τ) be a topological space such that τ is finite. Let $O = \{A_\alpha : \alpha \in \Delta\}$ be any open cover for X . Since $O \subseteq \tau$, any element of O is an element of τ . And since τ is finite, O must be of the form

$$O = \{A_{\alpha_1}, A_{\alpha_2}, A_{\alpha_3}, \dots, A_{\alpha_n}, A, A, \dots\}.$$

This implies that the subfamily

$$O_1 = \{A_{\alpha_1}, A_{\alpha_2}, A_{\alpha_3}, \dots, A_{\alpha_n}, A\}$$

of O covers X . Hence X is compact.

Definition 1.5: A topology is called a chain-element topology if it is an element of a family of topologies which form a pairwise comparable (decreasing or increasing) chain of topologies on a set.

Remark: The following results have been proved in Alexander and Chika (2025). A reader who wants to see the proof or examples in these results may consult the cited work.

Lemma 1.2 (The Cofinite Topology Lemma): Let A and B be two infinite subsets of an infinite set X such that $A \subset B$ (where A is a proper subset of B). Then there exist semi-cofinite topologies T_{CA} and T_{CB} on X , induced respectively by A and B such that $T_{CA} < T_{CB}$; that is, T_{CA} is strictly weaker than T_{CB} . This result also holds if X is finite.

Lemma 1.3: If Y is a nonempty proper subset of X , then Y induces a semi-cofinite topology, say T_{CY} on X .

Theorem 1.1 (Cofinite Topology Theorem): Let X be any infinite set. There exists a sequence $\{\tau_1, \tau_2, \tau_3, \dots\}$ of semi-cofinite topologies on X , forming a chain in that

$$C = T_{CX} > \tau_1 > \tau_2 > \tau_3 > \dots$$

where $C = T_{CX}$ is the cofinite topology on X .

Theorem 1.2 (Branching): Each of the chain element topologies under the cofinite topology theorem is itself at the peak of yet another chain of (semi-cofinite) topologies. If the original set X is infinite, then this branching will be endless; if X is finite, the branching will terminate.

2. Main Results—Proliferation of Compactness

Remark: In our referenced works Seymour (1965) and James (2000) it can be seen that the idea of a cofinite topological space being a compact space has been exposed. Hence we do not claim the next theorem (2.1) as completely our innovation, but rather as an existing lemma that we intend to use in the proof of our innovative results. All other results in this section are our innovation.

Theorem 2.1: Let (X, τ) be a cofinite topological space. Then any subset of X is compact.

Proof: Let Y be any subset of X . Let $\Psi = \{A_\alpha : \alpha \in \Delta\}$ be an open cover for Y . Let $A_0 \in \Psi$ be an arbitrarily chosen but fixed element of Ψ . Then since we are in a cofinite topological space, the complement of A_0 is finite, say $A_0^c = \{x_1, \dots, x_n\}$, and since certainly $X \subseteq A_0 \cup A_0^c$ and $Y \subset X$, it follows that $Y \subseteq A_0 \cup A_0^c$. Then, since Ψ is an open cover for Y and $Y \subseteq A_0 \cup A_0^c$, there exists $A_1, \dots, A_n \in \Psi$

such that $A_0^c \subset A_1 \cup \dots \cup A_n$. Therefore $Y \subseteq A_0 \cup A_0^c$ implies that

$$Y \subseteq A_0 \cup A_0^c \subseteq A_0 \cup A_1 \cup \dots \cup A_n = \bigcup_{i=0}^n A_i.$$

That is, the arbitrary open cover Ψ for Y has a finite open sub-cover for Y . It follows from definition 1.3 that Y is compact.

Note: If no element of $A_0^c = \{x_1, \dots, x_n\}$ belongs to Y , then since $Y \subset X \subseteq A_0 \cup A_0^c$, it follows that $Y \subseteq A_0$. This means that the open cover Ψ for Y has an easy finite open sub-cover $\Psi^* = \{A_0\}$ for Y . The same conclusion follows if $A_0^c = \emptyset$, the empty set.

Example 1: Let $X = \mathbb{R}$, the set of real numbers with the cofinite topology, $Y = (0, 1)$ a sub-interval of \mathbb{R} , and let $A_0 = \mathbb{R} - \{2, 3\}$. Then $A_0^c = \{2, 3\}$ and it is clear that Y does not intersect A_0^c ; namely $Y \cap A_0^c = \emptyset$. From this we see that any open cover Ψ for Y containing A_0 has $\Psi^* = \{A_0\}$ as a finite open sub-cover for Y .

Example 2: Let $X = \mathbb{R}$, the set of real numbers with the cofinite topology, $Y = (0, 5)$ a sub-interval of \mathbb{R} , and let $A_0 = \mathbb{R} - \{2, 3\}$. Then and it is clear that (now) Y does intersect A_0^c ; namely $Y \cap A_0^c = \{2, 3\}$. From this we see that if Ψ is an open cover for Y containing A_0 , then any finite open sub-cover Ψ^* of Ψ , for Y , must contain a set (or sets) that includes the numbers 2 and 3.

Remark: We should observe that the sets Y in examples 1 and 2 are not compact when the set \mathbb{R} of real numbers is endowed with its usual topology.

Example 3: Let X be any nonempty set (finite, infinite, countable or uncountable) with the cofinite topology, and let $Y = X - \{x_1\}$. Then any open cover Ψ for Y will have a finite sub-cover Ψ^* whose elements would not have to contain the point x_1 . In other words, any subfamily Ψ^* of Ψ containing sets of finite complement which do not contain the point x_1 will be an open subcover for Y . For example, if we let $A = X - \{x_1, x_2\}$ and $B = X - \{x_1, x_3\}$, then $\Psi^* = \{A, B\}$ is a finite open subcover for Y .

Theorem 2.2: Let (X, τ_s) be a semi-cofinite topological space. Then any subset of X is compact.

Proof: Let Y be a subset of X and let $\Psi = \{A_\alpha : \alpha \in \Delta\}$ be any open cover for Y . Let $A_0 \in \Psi$ be an arbitrarily chosen but fixed element of Ψ . Then since we are in a semi-cofinite topological space, the complement of A_0 is finite, say $A_0^c = \{x_1, \dots, x_n\}$. Because certainly $X \subseteq A_0 \cup A_0^c$, and $Y \subset X$, it follows that $Y \subseteq A_0 \cup A_0^c$.

If no element of is contained in Y , then the last inclusion means that $Y \subset A_0 \in \Psi$; hence the open cover Ψ for Y has a finite subcover $\Psi^* = \{A_0\}$.

Suppose m elements of are contained in Y , where $1 \leq m \leq n$. Then, since Ψ is an open cover for Y and $Y \subseteq A_0 \cup A_0^c$, let $A_1, \dots, A_m \in \Psi$ be the elements of Ψ containing the elements of $Y \cap A_0^c$. Then it is clear that $\Psi^* = \{A_0, A_1, \dots, A_m\} \in \Psi$ and that $Y \subseteq A_0 \cup A_1 \cup \dots \cup A_m$. Therefore we have that

$$Y \subseteq A_0 \cup A_1 \cup \dots \cup A_m = \bigcup_{i=0}^m A_i$$

That is, the arbitrary open cover Ψ for Y has a finite open sub-cover for Y .

Therefore Y is compact.

Note: The following two examples represent examples 1 and 2 of our paper already referenced above.

Example 4: Let $N = \{0, 1, 2, \dots\}$ denote the set of natural numbers. For each $n \in N$ let G_n be the set of all real numbers excluding the first n natural numbers. Thus for instance

$$G_0 = R - \{\} = R;$$

$$G_1 = R - \{0\};$$

$$G_2 = R - \{0, 1\};$$

$$G_3 = R - \{0, 1, 2\};$$

...

$$G_n = R - \{0, 1, 2, 3, \dots, n-1\}$$

$$\text{Let } T_{CN} = \{\emptyset, G_n\}_{n \in N}$$

Then it is easy to see that

1. The empty set is in T_{CN} , from the way T_{CN} is defined.
2. The whole set R of real numbers is in T_{CN} .
3. The complement G_n^c of every set in T_{CN} , apart from the empty set, is finite; precisely G_n^c contains the first n natural numbers.
4. And that T_{CN} is closed under finite intersections and arbitrary unions.
5. Hence T_{CN} is a topology on R , satisfying all but one property of the cofinite topology, on R , namely that it is not the family of *all* subsets of R whose complements are finite, together with the empty set. Hence T_{CN} is an example of what we call semi-cofinite topology, on the set R of real numbers. It is strictly weaker than the cofinite topology on R .

We shall call T_{CN} the semi-cofinite topology on R generated by the set N of natural numbers. Any subset of R is compact in this topological space.

Example 5: Let $Z = \{0, 1, -1, 2, -2, 3, -3, \dots\}$ denote the set of all integers, arranged thus. For each $n \in N = \{0, 1, 2, \dots\}$, let G_n be R without the first n integers under the arrangement thus made of Z . Hence for instance

$$G_0 = R;$$

$$G_1 = R - \{0\};$$

$$G_2 = R - \{0, 1\};$$

$$G_3 = R - \{0, 1, -1\};$$

$$G_4 = R - \{0, 1, -1, 2\};$$

$$G_5 = R - \{0, 1, -1, 2, -2\};$$

.

.

.

...

etc.

Then $T_{CZ} = \{\emptyset, G_n\}_{n \in N}$ is a semi-cofinite topology (different from the one above) on R , as can easily be verified.

We observe that in T_{CN} the set $G_2 = R - \{0, 1\}$ and that in T_{CZ} $G_2 = R - \{0, 1\}$. However in T_{CN} $G_3 = R - \{0, 1, 2\}$ and in T_{CZ} $G_3 = R - \{0, 1, -1\}$ and we see that although G_2 is common to both T_{CN} and T_{CZ} , G_3 is not common to the two

topologies on R . It may be further verified that G_n is not common to the two topologies if $n \geq 3$. Hence these two topologies are not comparable.

But we can obtain two comparable semi-cofinite topologies on R based on the two subsets N and Z . Let Z be written in the alternative (and usual) form $Z = \{\dots, -2, -1, 0, 1, 2, \dots\}$ and let $G_n = R - \{n \text{ integers}\}$ for each $n \in N = \{0, 1, 2, \dots\}$. That is, G_n is R without a finite number of whole numbers. We can also observe that the complement of each G_n is a finite number of integers. This observation is helpful in proving that the family $T_Z = \{\emptyset, G_n\}_{n \in N}$ is closed under arbitrary unions; for if $\{G_\alpha\}_{\alpha \in \Delta} \subset T_Z$ is any family of sets in T_Z , then

$$\bigcap_{\alpha \in \Delta} G_\alpha^c$$

is a finite number of integers. It follows from De Morgan's law that the set

$$\left(\bigcap_{\alpha \in \Delta} G_\alpha^c \right)^c = \bigcup_{\alpha \in \Delta} G_\alpha$$

is R without n whole numbers. Similarly, we see that

$$\bigcap_{i=1}^n G_i = \left(\bigcup_{i=1}^n G_i^c \right)^c$$

is the complement of a finite number of whole numbers. Hence T_Z is a semi-cofinite topology on R . And it is easy to see that T_{CN} is strictly weaker than T_Z . There are other ways of constructing strictly comparable pairs of semi-cofinite topology on any infinite set. Several developments in our earlier article (referenced) show that, see (Alexander and Chika, 2025). Also any subset of the set R of real numbers is compact in this topological space.

Remark: Let X be any nonempty set. The cofinite topology on X and all the semi-cofinite topologies on X can be seen as the topologies on X with *largest open sets*. The open sets of these topologies make any subset of X look small (compact) so that few sets (a finite number) from any open cover can cover each *openly covered* subset. This can serve as another useful way of seeing compactness.

Theorem 2.3 (Proliferation of Compactness 1): Any nonempty set X with cardinality $n \geq 2$ has as many topologies as at least the cardinality of X , each of which makes X compact.

Proof: Because any proper subset of a set X can induce a unique semi-cofinite topology on X , and because (by theorem 2.2) each semi-cofinite topology is a compact topology on X , we can use the elements of X to index such compact topologies on X . The proof of this theorem is complete.

Example 6: Let $X = \{x_1, \dots, x_{10}\}$, and $X_1 = X - \{x_1\} = \{x_2, \dots, x_{10}\}$, and $X_2 = X - \{x_2\} = \{x_1, x_3, x_4, \dots, x_{10}\}, \dots, X - \{x_{10}\} = \{x_1, \dots, x_9\}$. Then by lemma 1.3 each ($i = 1, \dots, 10$) induces a semi-cofinite topology T_{CX_i} on X . If we add the cofinite topology of X to the list, then we would have exactly 11 topologies on X , each of which by theorems 2.1 and 2.2 makes not only X but all its subsets compact.

Example 7: It is known that if the cardinality of a set X is n , then X has exactly 2^n subsets. We can remove the family $\{\emptyset, X\}$ from these 2^n subsets of X , and have remaining only $2^n - 2$ subsets, for $n \geq 2$. Then since for all natural numbers $n \geq 2$, we know that $2^n - 2 \geq n$, lemma 1.3 and theorems 2.1 and 2.2 imply that X has as many compact topologies as at least the cardinality of X . In example 6 above, X has at least as many compact topologies as $2^{10} - 2$.

Theorem 2.4 (Proliferation of Compactness 2): If $Card(X) = n$, then there exists a chain

$$C = \{\tau_\alpha; \alpha \in \Delta\}$$

of pairwise comparable compact topologies on X such that $Card(\Delta) = Card(C) \geq n$.

Proof: A pairwise comparable chain

$$C = \{\tau_\alpha; \alpha \in \Delta\}$$

of topologies on X exists, by lemma 1.2 and theorem 1.1. By theorem 2.3, the topologies in the chain C are compact topologies and we can choose only a number of them so that $Card(\Delta) = Card(C) \geq n$. Observe that we can index the

chain C by the elements of X , so that we just write $C = \{\tau_x : x \in X\}$. This notation more directly helps us to see the link between the cardinality of C and that of the indexing set Δ . This proof is complete.

Corollary 2.1: *For any pair (A, x) of an element $x \in X$ and a subset $A \subset X$, there exists a topology τ_x on X with respect to which not only A is compact but all other subsets of X are compact.*

Theorem 2.5 (Proliferation of Compactness 3—Branching): *Each of the chain element compact topologies in theorem 2.4 is itself at the peak of yet another chain of pairwise comparable compact topologies. If the original set X is infinite, then this branching will be endless; if X is finite, the branching will terminate.*

Proof: This is the collective implication of lemmas 1.1, 1.2, 1.3 and theorems 1.1, 1.2, 2.1, 2.2, 2.3 and 2.4.

3. Summary and Conclusions

1. From our earlier results in the reference (Alexander and Chika, 2025), the cofinite topology of any nonempty set X is a tree of many branches of semi-cofinite topologies.
2. From theorems 2.2 to 2.5 (combined with the pre-existing theorem 2.1) here, we see that any nonempty set X is (topologically speaking) a tree of many branches of pairwise comparable compact topologies.
3. From theorems 2.1 and 2.2 all subsets of X are compact with respect to each of these—probably infinitely many—compact topologies on the set X . In other words, a proliferation of compactness exists for any nonempty set X . Therefore compactness should no more be seen as a rare property to be possessed by any set.
4. If A is a subset of a nonempty set X , then there exist as many topologies, on X , as the cardinality of X , with respect to which the subset A is compact (Theorem 2.4). In short, by corollary 2.1 for any pair (A, x) of an element $x \in X$ and a subset $A \subset X$, there exists a topology τ_x on X with respect to which not only A is compact but all other subsets of X are compact. This particular result will aid and encourage analysis in a way that we cannot now comprehensively tell for future. It calls for further research.

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