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On the Order Estimates for Specific Functions of $\zeta(s)$ and its Contribution towards the Analytic Proof of the Prime Number Theorem

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Abstract

This article provides a proof of the famous Prime Number Theorem by establishing an analogous statement of the same in terms of the second Chebyshev Function $\psi(x)$. We shall be extensively using complex analytic techniques in addition to certain meromorphic properties of the Reimann Zeta Function $\zeta(s)$ and its Analytic Continuation Property a priori using Riemann's Functional Equation in order to establish our desired result.

Keywords: *Prime number theorem, Riemann zeta function, Holomorphic, Contour integral, Critical line*

1. Introduction

In Number Theory, the Prime Number Theorem (PNT) describes the asymptotic distribution of the prime numbers among the positive integers. It formalizes the intuitive idea that primes become less common as they become larger by precisely quantifying the rate at which this occurs. The theorem was proved independently by *Jacques Hadamard and Charles Jean de la Vallee-Poussin* in 1896 using ideas introduced by Bernhard Riemann (in particular, the *Riemann zeta function*).

The first such distribution found is $\pi(x) \sim \dfrac{x}{\log x}$, where $\pi(x)$ is the prime-counting function and $\log x$ is the natural logarithm of x. This means that for large enough x, the probability that a random integer n not greater than x is prime is very close to $\dfrac{1}{\log x}$. Consequently, a random integer with at most $2k$ digits (for large enough k) is about half as likely to be prime as a random integer with at most k digits. For example, among the positive integers of at most 1000 digits, about one in 2300 is prime ($\log 10^{1000} \approx 2302.6$), whereas among positive integers of at most 2000 digits, about one in 4600 is prime ($\log 10^{2000} \approx 4605.2$). In other words, the average gap between consecutive prime numbers among the first n integers is roughly $\log n$.

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2. Statement of the Prime Number Theorem

2.1. Notion of Arithmetic Functions

We first introduce some special arithmetic functions and notations required to appreciate the analytic aspects of the Prime Number Theorem (Jameson, 2003).

Definition 2.1.1.: For each $x \geq 0$, we define,

$\pi(x) :=$ The number of primes $\leq x$.

Definition 2.1.2.: For each $x \geq 0$, we define,

$$\psi(x) := \sum_{n \leq x} \Lambda(n),$$

Where,

$$\Lambda(n) := \begin{cases} \log p & \text{if } n = p^m, p^m \leq x, m \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases} \qquad \text{...(2.1)}$$

$\Lambda(n)$ is said to be the "*Mangoldt Function*".

Therefore,

$$\psi(x) = \sum_{n \leq x} \Lambda(n) = \sum_{m=1}^{\infty} \sum_{p, p^m \leq x} \Lambda(p^m) = \sum_{m=1}^{\infty} \sum_{p \leq x^{\frac{1}{m}}} \log p \qquad \text{...(2.2)}$$

Definition 2.1.3.: (Chebyshev Theta Function) For each $x \geq 0$, we define,

$$\vartheta(x) := \sum_{p \leq x} \log p$$

Remark 2.1.1.: A priori from definitions (2.1.2) and (2.1.3), it can be deduced that,

$$\psi(x) = \sum_{m \leq \log_2 x} \vartheta\left(x^{\frac{1}{m}}\right)$$

Definition 2.1.4.: (M Nobius Function) The Mobius Function μ is defined as follows:

$\mu(1) = 1$.

If $n > 1$, such that suppose, $n = p_1^{a_1} p_2^{a_2} p_3^{a_3} \cdots p_k^{a_k}$. Then,

$$\mu(n) := \begin{cases} (-1)^k & \text{if } a_1 = a_2 = \ldots = a_k = 1 \\ 0 & \text{otherwise} \end{cases} \qquad \text{...(2.3)}$$

Definition 2.1.5.: (Big O Notation) Given $g(x) > 0 \ \forall x \geq a$, the notation, $f(x) = O(g(x))$ implies that, the quotient, $\dfrac{f(x)}{g(x)}$

is bounded for all $x \geq a$; i.e., $\exists$ a constant $M > 0$ such that,

$$\left| f(x) \right| \leq M \cdot g(x), \ \forall x \geq a$$

2.2. Asymptotic Bound for π(x)

Definition 2.2.1.: We say $f(x)$ is asymptotic to $g(x)$, and denote it by, $f(x) \sim g(x)$ if, $\lim\limits_{x \to \infty} \dfrac{f(x)}{g(x)} = 1$.

A priori having all the necessary notations and definitions in our arsenal, we can formally state the "Prime Number Theorem".

Theorem 2.2.1.: *(Prime Number Theorem) (George, 1971; Ireland and Michael, 1990). For every real number $x \geq 0$, we define the prime counting function π(x) as in (2.1.1). Then, the following estimate is valid.*

$$\pi(x) \sim \frac{x}{\log x} \qquad\qquad ...(2.4)$$

Equivalently, (Andrew, 2008),

$$\lim_{x \to \infty} \frac{\pi(x) \log x}{x} = 1 \qquad\qquad ...(2.5)$$

Given the difficulty in establishing the above result in support of proving the prime number theorem, we intend to explore about the possibility of working on an equivalent version of Theorem (2.2.1), which obviously will be less tedious and easy to comprehend.

Theorem 2.2.2.: *We have,*

$$\vartheta(x) \sim \pi(x) \log x \qquad\qquad ...(2.6)$$

Proof: A priori it follows from definition and the Theorem of Partial Sums of Dirichlet Product that,

$$\vartheta(x) = \sum_{x^{1-\varepsilon} \leq p \leq x} \log p \left\lfloor \frac{\log x}{\log p} \right\rfloor \leq \sum_{p \leq x} \log x \leq \pi(x) \log x \qquad\qquad ...(2.7)$$

Hence, for arbitrary $\varepsilon > 0$,

$$\vartheta(x) = \sum_{x^{1-\varepsilon} \leq p \leq x} \log p \geq \sum_{x^{1-\varepsilon} \leq p \leq x} (1-\varepsilon) \log x = (1-\varepsilon)\left(\pi(x) + O\left(x^{1-\varepsilon}\right)\right) \log x$$

Therefore, we conclude,

$$\Rightarrow \vartheta(x) \to \pi(x) \log x, \text{ as } x \to \infty \qquad\qquad ...(2.8)$$

$$\Rightarrow \vartheta(x) \sim \pi(x) \log x \qquad\qquad ...(2.9)$$

An application of Theorem (2.2.2) along with properties of the functions $\vartheta(x)$ and $\psi(x)$ we can indeed formulate an alternative statement of Prime Number Theorem:

Theorem 2.2.3.:

$$\psi(x) \sim x \text{ as } x \to \infty \qquad\qquad ...(2.10)$$

Hence, in order to establish the Prime Number Theorem, it only suffices to show that, statement (2.2.3) hods true.

3. A Brief Survey on Complex Analysis

In this section, we shall recall some important results in Complex Analysis, pertinent to the proof. Readers are encouraged to refer to (Elias and Rami, 2003) for further explanations in these topics.

Theorem 3.0.1.: *(Riemann-Lebesgue Lemma) If f is L^1 integrable on $\mathbb{R}^d$, i.e., to say, if the Lebesgue integral of |f| is finite, then the Fourier transform of f satisfies,*

$$\hat{f}(z) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i z \cdot x} dx \to 0 \ as \ |z| \to \infty$$

Theorem 3.0.2.: *(Cauchy Integral Theorem) In mathematics, the Cauchy Integral Theorem (also known as the Cauchy-Goursat theorem) in complex analysis, named after Augustin-Louis Cauchy, is an important statement about line integrals for holomorphic functions in the complex plane. Essentially, it says that if two different paths connect the same two points, and a function is holomorphic everywhere "in between" the two paths, then the two path integrals of the function will be the same.*

The theorem is usually formulated for closed paths as follows:

Let U be an open subset of $\mathbb{C}$ which is simply connected, let *f*: $U \to \mathbb{C}$ be a holomorphic function, and let γ be a rectifiable path in U whose start point is equal to its end point. Then,

$$\oint_{\gamma} f(z) dz = 0$$

A precise (*homology*) version can be stated using *winding numbers*. The winding number of a closed curve around a point a not on the curve is the integral of $\dfrac{f(z)}{2\pi i}$, where $f(z) = \dfrac{1}{(z-a)}$ around the curve. It is an integer. Briefly, the path integral along a Jordan curve of a function holomorphic in the interior of the curve, is 0. Instead of a single closed path we can consider a linear combination of closed paths, where the scalars are integers. Such a combination is called a closed chain, and one defines an integral along the chain as a linear combination of integrals over individual paths. A closed chain is called a cycle in a region, if it is homologous to 0 in the region; that is, the winding number, expressed by the integral of $\dfrac{1}{(z-a)}$ over the closed chain, is 0 for each point *a* not in the region. This means that the closed chain does not wind around points outside the region. Then Cauchy's Theorem can be stated as the integral of a function holomorphic in an open set taken around any cycle in the open set is 0. An example is furnished by the ring-shaped region. This version is crucial for rigorous derivation of *Laurent Series and Cauchy's Residue Formula* without involving any physical notions such as cross cuts or deformations. This version enables the extension of Cauchy's theorem to multiply-connected regions analytically.

Theorem 3.0.3.: *(Cauchy Residue Theorem) In complex analysis, the Residue Theorem, sometimes called Cauchy's Residue Theorem, is a powerful tool to evaluate line integrals of analytic functions over closed curves; it can often be used to compute real integrals as well. It generalizes the Cauchy Integral Theorem and Cauchy's Integral Formula. From a geometrical perspective, it is a special case of the generalized Stokes' Theorem. The statement goes as follows:*

Suppose U is a simply connected open subset of the complex plane, and $a_1, a_2, ..., a_n$ are finitely many points of U and *f* is a function which is defined and holomorphic on $U/a_1, a_2, ..., a_n$. If γ is a closed rectifiable curve in U which does not meet any of the a_k,

$$\oint_{\gamma} f(z) dz = 2\pi i \sum_{k=1}^{n} \eta(\gamma, a_k) \operatorname{Res}(f; a_k)$$

If γ is a positively oriented simple closed curve, $\eta(\gamma, a_k) = 1$ if a_k is in the interior of γ, and 0 otherwise, so,

$$\oint_{\gamma} f(z) dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f; a_k)$$

with the sum over those *k* for which ak is inside γ.

Here, $Res(f; a_k)$ denotes the residue of f at a_k, and $\eta(\gamma, a_k)$ is the winding number of the curve γ about the point a_k. This winding number is an integer which intuitively measures how many times the curve γ winds around the point a_k; it is positive if γ moves in a counter clockwise (mathematically positive) manner around a_k and 0 if γ doesn't move around a_k at all.

4. Some Important Results

Lemma 4.0.1.: Given any arithmetic function $a(n)$, suppose,

$$A(x) = \sum_{n \leq x} a(n),$$

where, $A(x) = 0$, if $x < 1$. Then,

$$\sum_{n \leq x} (x - a) a(n) = \int_1^x A(t) dt \qquad \qquad ...(4.1)$$

We apply Abel's Identity to prove the result.

Theorem 4.0.2.:

$$\sum_{n \leq x} a(n) f(n) = A(x) f(x) - \int_1^x A(t) f'(t) dt \qquad \qquad ...(4.2)$$

provided f has a continuous derivative on $[1, x]$.

Proof: Choosing $f(t) = t$, we obtain,

$$\sum_{n \leq x} a(n) f(n) = \sum_{n \leq x} n a(n) \text{ and } A(x) f(x) = n \sum_{n \leq x} a(n)$$

Thus the result follows from *Theorem* (4.2).

In order to estimate $A(x)$ as in above, we require the following lemma can also be regarded as a form of L'H'ospital's rule for increasing piece-wise linear functions.

Lemma 4.0.3.: *Let* $A(x) = \sum_{n \leq x} a(n)$ *and let* $A_1(x) = \int_1^x A(t) dt$ *. Assume also that,* $a(n) \geq 0, \ \forall n \in \mathbb{N}$ *. Given the asymptotic relation,*

$$A_1(x) \sim Lx^c \text{ as } x \to \infty \qquad \qquad ...(4.3)$$

For some c > 0 and L > 0, we shall have,

$$A(x) \sim cLx^{c-1} \text{ as } x \to \infty \qquad \qquad ...(4.4)$$

In other words, formal differentiation of (4.3) gives the result (4.4).

Proof: The function $A(x)$ is increasing, since the $a(n)$'s are non-negative. Consider any $\beta > 1$.

Thus, we get,

$$A_1(\beta x) - A_1(x) = \int_x^{\beta x} A(u) du \geq \int_x^{\beta x} A(x) du = A(x)(\beta(x) - x) = x(\beta - 1) A(x)$$

This implies,

$$xA(x) \leq \frac{1}{(\beta-1)} A_1(\beta x) - A_1(x)$$

or,

$$\frac{A(x)}{x^{c-1}} \leq \frac{1}{(\beta-1)} \left\{ \frac{A_1(\beta x)}{(\beta x)^c} \beta^c - \frac{A_1(x)}{x^c} \right\}$$

Keeping β fixed and taking $x \to \infty$ in (4.5), we obtain,

$$\limsup_{x \to \infty} \frac{A(x)}{x^{c-1}} \leq \frac{1}{\beta-1} (L\beta^c - L) = L\frac{\beta^c - 1}{\beta-1} \qquad \text{...(4.5)}$$

Further, consider $\beta \to 1^+$. The quotient on the right is the difference quotient for the derivative of x^c at $x = 1$ and has the limit c. Therefore,

$$\limsup_{x \to \infty} \frac{A(x)}{x^{c-1}} \leq cL \qquad \text{...(4.6)}$$

Assume any α with $0 < \alpha < 1$ and consider the difference, $\{A_1(x) - A_1(\alpha x)\}$. By similar arguments done previously during the proof of this lemma, we get,

$$\liminf_{x \to \infty} \frac{A(x)}{x^{c-1}} \geq L\frac{1-\alpha^c}{1-\alpha} \qquad \text{...(4.7)}$$

Now, as $\alpha \to 1^-$, the term in the R.H.S. tends to cL. This, together with (4.6) evokes that,

$$\frac{A(x)}{x^{c-1}} \to cL \text{ as } x \to \infty \qquad \text{...(4.8)}$$

And we are done.

A priori if we assume, $a(n) = \Lambda(n)$, then $A(x) = \psi(x)$, $A_1(x) = \psi_1(x)$, and $a_n \geq 0$.

It follows from lemmas (4.0.1) and (4.0.3),

Theorem 4.0.4.:

$$\psi_1(x) = \sum_{n \leq x} (x-n)\lambda(n) \qquad \text{...(4.9)}$$

Where, the Liouville's Function, $\lambda(n)$ is defined as follows:

$$\lambda(n) = \begin{cases} 1 & \text{if } n = 1 \\ (-1)^{\sum_{i=1}^{k} \alpha_i} & \text{if } n = \prod_{i=1}^{k} p_i^{\alpha_i} \end{cases}$$

Also the asymptotic relation, $\psi_1(x) \sim \frac{x^2}{2}$ implies, $\psi(x) \sim x$ as $x \to \infty$.

Our next goal is to express $\dfrac{\psi_1(x)}{x^2}$ as a Contour Integral involving the zeta function (We shall provide an overview of the Riemann Zeta Function in the next section). For this, we will require the special cases, $k = 1$ and $k = 2$ of the following lemma on contour integrals.

Lemma 4.0.5.: If $c > 0$ and $u > 0$, then for every integer $k \geq 1$, we have,

$$\frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} \frac{u^{-z}}{z(z+1)(z+2)\cdots(z+k)}\,dz = \begin{cases} \dfrac{1}{k!}(1-u)^k & \text{if } 0 < u \leq 1 \\ 0 & \text{if } u > 1 \end{cases}$$

the integral being absolutely convergent.

Proof: We observe that, the integrand is $= \dfrac{u^{-z}\Gamma(z)}{\Gamma(z+k+1)}$. (This follows by repeatedly using the functional equation, $\Gamma(z + 1) = z.\Gamma(z)$), and application of *Cauchy's Residue Theorem*. (For detailed proof of the lemma, readers can see (Tom, 1976).

5. Analytic Continuation Property of $\zeta(s)$

5.1. Motivation to Study $\zeta(s)$

Riemann Zeta Function (Tom, 1976) is an integral part of *Analytic Number Theory* (Jean-Marie and Florian, 1948; Henry, 2007), often treated as a special case of the *Hurwitz Zeta Function* $\zeta(s, a)$, defined for $Re(s) > 1$, as the series,

$$\zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}, \text{ where, } a \in \mathbb{R}, \ 0 < a \leq 1 \text{ is fixed.}$$

Riemann Zeta Function was first introduced by Leonhard Euler in the first half of eighteenth century, using only Real numbers. Also, he even computed the values of the zeta function at even positive integers.

Later on, famous mathematician Bernhard Riemann extended Euler's definition of the Riemann Zeta Function on $\mathbb{R}$ to the field of Complex Numbers, also deriving the meromorphic continuation and functional equation, and gave us an idea about the relationship between the zeroes of the Riemann Zeta Function and the distribution of prime numbers. The details can be found in his article titled On the Number of Primes less than a given Magnitude published in 1859. That's the reason why, afterwards, this function was named after him.

Riemann Zeta Function is hugely significant in the field of Number Theory. For example, $\zeta(2)$ provides solution to the famous "Basel Problem". Also, famous Greek-French mathematician Roger Apery proved in the year 1979 that, $\zeta(3)$ is irrational. Euler established the fact that, the Riemann Zeta Function yields rational values at the negative integer points and moreover, these values are incredibly useful in the field of Modular Forms. Functions like, Dirichlet Series, Dirichlet L-functions, and L-functions are often considered to be generalisation of the Riemann Zeta Function.

5.2. Analytic Properties of $\zeta(s)$

Enough with its historical significance, let us give a proper definition of the *Riemann Zeta Function* (Ivan *et al.*, 1991; Alan, 2012; Newman, 1980; Grosswald, 1964) below.

Definition 5.2.1.: (Riemann Zeta Function) The Riemann Zeta Function $\zeta(s)$ is defined as,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \text{ where, } s \in \mathbb{C}, \text{ with } Re(s) > 1$$

And we can further extend the Riemann Zeta Function to the whole complex plane using the Analytic Continuation Property of the function defined for $Re(s) > 1$.

A priori from the definition of the Gamma Function $\Gamma(s)$ given as,

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} dx, \text{ where, } s \in \mathbb{C}, \ Re(s) > 0$$

we can thus provide an alternative definition of the Riemann Zeta Function as,

Definition 5.2.2.: We have,

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx, \text{ where, } s \in \mathbb{C} \text{ and } Re(s) > 1$$

As for some important properties of $\zeta(s)$, they can be summarized as follows:

Proposition 5.2.1.: $\zeta(s)$ is meromorphic on $\mathbb{C}$, i.e., $\zeta(s)$ is holomorphic everywhere except for a simple pole at $s = 1$ with residue 1.

Proof: Follows from the deduction that,

$$\lim_{s \to 1} (s-1) \cdot \zeta(s) = 1$$

Proposition 5.2.2.: *For s = 1, $\zeta(s)$ is the harmonic series that diverges to $+\infty$.*

Proposition 5.2.3.: *(Basel Problem)*

$$\zeta(2) = \sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$$

Proposition 5.2.4.: *(Trivial Zero-Free Region) The Riemann Zeta Function $\zeta(s)$ has no zeroes in the region,* $\{ s \in \mathbb{C} : Re(s) > 1 \}$.

5.3. Euler Product Representation of $\zeta(s)$

Deduced by famous mathematician Euler, in the year 1737, the following identity establishes a relation between the Riemann Zeta Function and prime numbers. The result is as follows:

Theorem 5.3.1.: *(Euler Product Formula) We have,*

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s} = \prod_{p=prime} \frac{1}{(1 - p^{-s})}$$

Where, the product on the R.H.S. is taken over all primes p and, converges for $Re(s) > 1$.

Proof: We shall state a lemma in support of establishing the above result (Gerald, 1984).

Lemma 5.3.2.: Suppose $\{a_n\}_{n \in \mathbb{N}}$ be a sequence of complex numbers with $a_n \neq -1, \ \forall n \in \mathbb{N}$.

Then,

$$\sum_{n=1}^\infty |a_n| \text{ is concergent, implies, } \prod_{n=1}^\infty (1 + a_n) \text{ is convergent}$$

Assume, $\sigma = Re(s)$. Then for natural numbers N_1 and N_2 with, $N_1 < N_2$ (without loss of generality),

$$\left| \sum_{N_1+1}^{N_2} \frac{1}{n^s} \right| \leq \left| \sum_{N_1+1}^{N_2} \frac{1}{n^\sigma} \right| \left(\text{Using the result, } |n^{-\sigma}| = |n^{-s}| \right)$$

A priori using the fact,

$$n^{-\sigma} \le \int_{n-1}^{n} x^{-\sigma} dx$$

performing summation on both sides yield,

$$\sum_{N_1+1}^{N_2} n^{-\sigma} \le \int_{N_1}^{N_2} x^{-\sigma} dx = \frac{1}{\sigma} \left(N_1^{-\sigma} - N_2^{-\sigma} \right) \to 0 \text{ as, } N_1, N_2 \to \infty$$

Hence, $\sum_{n=1}^{\infty} n^{-s}$ is convergent. (Applying Cauchy's Criterion for convergence of series)

Also,

$$\sum_{p=prime} \left| p^{-s} \right| = \sum_{p=prime} p^{-\sigma} \le \sum_{n=1}^{\infty} n^{-\sigma}.$$

Hence, we conclude that, $\sum_{p=prime} p^{-s}$ is absolutely convergent.

Applying Lemma (5.3.2), we obtain that, the product, $\prod_{p=prime} \left(1 - p^{-s} \right)$ is absolutely convergent, hence convergent.

Thus,

$$\prod_{p=prime} \left(1 - p^{-s} \right) \text{ is also convergent.}$$

To prove that, both sides of the given identity in this theorem are equal, we observe that,

$$\prod_{p \le N_2} \frac{1}{\left(1 - \chi(p) p^{-s} \right)} = \sum_{n > N_2} \chi(n) n^{-s} + \sum_{n \le N_2} \chi(n) n^{-s}$$

As, $N_2 \to \infty$,

R.H.S. $= \sum_{n \in \mathbb{N}} n^{-s}$ (The sum being absolutely convergent)

Therefore, the product on the L.H.S. tends to $\prod_{p=prime} \left(1 - p^{-s} \right)$, and hence, Euler's Formula is established.

5.4. Riemann's Functional Equation for ζ(s)

We define a particular form of Dirichlet Series $F(x, s)$ as,

$$F(x, s) = \sum_{n=1}^{\infty} \frac{e^{2\pi i n x}}{n^s}, \text{ where, } x \in \mathbb{R} \text{ and } Re(s) > 1$$

Important to observe that, $F(x, s)$ is a periodic function of x (also known as the periodic zeta function) with period 1 and, $F(1, s) = \zeta(s)$.

Theorem 5.4.1.: *(Hurwitz's Formula) We have,*

$$\zeta(1-s,a) = \frac{\Gamma(s)}{(2\pi)^s}\left\{e^{-\pi is/2}F(a,s) + e^{\pi is/2}F(-a,s)\right\}$$

For, $0 < a \le 1$ and $Re(s) > 1$.

Remark 5.4.2.: For $a \ne 1$, Hurwitz's Formula is valid for $Re(s) > 0$.

Using Hurwitz's Formula, we shall establish the functional equation for $\zeta(s)$.

Theorem 5.4.3.: *The functional equation for the Riemann Zeta Function $\zeta(s)$ is given by*

$$\zeta(1-s) = 2(2\pi)^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s)$$

In other words,

$$\zeta(s) = 2(2\pi)^{s-1}\Gamma(1-s)\cos\left(\frac{\pi s}{2}\right)\zeta(1-s)$$

Proof: Putting $a = 1$ in Hurwitz's Formula,

$$\zeta(1-s) = \frac{\Gamma(s)}{(2\pi)^s}\left\{e^{-\pi is/2}\zeta(s) + e^{\pi is/2}\zeta(s)\right\} = \frac{\Gamma(s)}{(2\pi)^s}2\cos\left(\frac{\pi s}{2}\right)\zeta(s)$$

Which proves the first part of the theorem. Replacing s by $1 - s$, we establish the equivalent definition of the functional equation for $\zeta(s)$.

Remark 5.4.4.: Putting, $s = 2n + 1$, $\forall n \in \mathbb{N}$, we obtain the trivial zeroes of $\zeta(s)$, consequently,

$\zeta(-2n) = 0 \ \forall n \in \mathbb{N}$.

6. Order Estimates for Specific Functions of $\zeta(s)$

6.1. A Contour Integral Representation of $\dfrac{\psi_1(x)}{x^2}$

Theorem 6.1.1.: *If $c > 1$ and $x \ge 1$, then,*

$$\frac{\psi_1(x)}{x^2} = \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i}\frac{x^{s-1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)ds \qquad \text{...(6.1)}$$

Proof: A priori from (4.9),

$$\frac{\psi_1(x)}{x} = \sum_{n \le x}\left(1 - \frac{n}{x}\right)\Lambda(n)$$

Apply Lemma (4.0.5) with putting $k = 1$, and $u = \dfrac{n}{x}$. In case when $n \le x$, we obtain,

$$\left(1 - \frac{n}{x}\right) = \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i}\frac{\left(\dfrac{x}{n}\right)^s}{s(s+1)}ds \qquad \text{...(6.2)}$$

Multiply (6.2) by $\Lambda(n)$ and summing over all $n \leq x$,

$$\frac{\psi_1(x)}{x} = \sum_{n \leq x} \frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} \frac{\Lambda(n)\left(\frac{x}{n}\right)^s}{s(s+1)} ds = \sum_{n=1}^{\infty} \frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} \frac{\Lambda(n)\left(\frac{x}{n}\right)^s}{s(s+1)} ds$$

Since the integral above vanishes if, $n > x$, we can derive that,

$$\frac{\psi_1(x)}{x} = \sum_{n=1}^{\infty} \int_{c-\infty i}^{c+\infty i} f_n(s) ds \text{ where, } 2\pi i f_n(x) = \Lambda(n)\frac{\left(\frac{x}{n}\right)^s}{s(s+1)} \qquad ...(6.3)$$

Suppose, we wish to interchange the sum and the integral in (6.3). For this it suffices to prove that, the series,

$$\sum_{n=1}^{\infty} \int_{c-\infty i}^{c+\infty i} |f_n(s)| ds \qquad ...(6.4)$$

is convergent. Important to observe that, the partial sum of the series given in (6.4) saitsfy the inequality,

$$\sum_{n=1}^{N} \int_{c-\infty i}^{c+\infty i} \frac{\Lambda(n)\left(\frac{x}{n}\right)^c}{|s||s+1|} ds = \sum_{n=1}^{N} \frac{\Lambda(n)}{n^c} \int_{c-\infty i}^{c+\infty i} \frac{x^c}{|s||s+1|} ds \leq A \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^c}$$

Where, A is a constant. Therefore, we can conclude that, (6.4) converges. Therefore, we can interchange the sum and the integral in (6.3) to obtain,

$$\frac{\psi_1(x)}{x} = \int_{c-\infty i}^{c+\infty i} \sum_{n=1}^{\infty} f_n(s) ds = \frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} \frac{x^s}{s(s+1)} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} ds$$

$$= \frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} \frac{x^s}{s(s+1)} \left(-\frac{\zeta'(s)}{\zeta(s)}\right) ds$$

Now, we divide both sides of the above identity to obtain the desired result.

Theorem 6.1.2.: *If $c > 1$ and $x \geq 1$ we have,*

$$\frac{\psi_1(x)}{x^2} - \frac{1}{2}\left(1 - \frac{1}{x}\right)^2 = \frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} x^{s-1} h(s) ds \qquad ...(6.5)$$

where,

$$h(s) = \frac{1}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1}\right) \qquad ...(6.6)$$

Proof: We use Lemma (4.0.5) with $k = 2$ to get,

$$\frac{1}{2}\left(1 - \frac{1}{x}\right)^2 = \frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} \frac{x^s}{s(s+1)(s+2)} ds \text{, where, } c > 0.$$

Replacing *s* by *s* – 1 in the integral (keeping *c* > 1),

$$\frac{1}{2}\left(1-\frac{1}{x}\right)^2 = \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i}\frac{x^{s-1}}{(s-1)(s)(s+1)}ds$$

Subtracting the above identity from the identity in Theorem (6.1.1), we get our desired result.

Remark 6.1.3.: If we parametrize the path of integration by writing $s = c + it$, we obtain, $x^{s-1} = x^{c-1}x^{it} = x^{c-1}e^{it\,\log x}$. As a result, Equation (6.5) becomes,

$$\frac{\psi_1(x)}{x^2}-\frac{1}{2}\left(1-\frac{1}{x}\right)^2 = \frac{x^{c-1}}{2}\int_{c-\infty i}^{c+\infty i}h(c+it)e^{it\log(x)}dt \qquad\qquad ...(6.7)$$

Our next task is to show that the R.H.S. of the identity (6.7) $\to 0$ as $x \to \infty$. As mentioned earlier, we first have to establish that, we can put $c = 1$ in (6.7). For this purpose, we need to study $\zeta(s)$ in the neighbourhood of the line, $\sigma = 1$ (taking, $s = \sigma + it$ in complex plane).

6.2. Upper Bounds for |ζ(s)| and |ζ'(s)| Near the Line σ= 1

In order to study $\zeta(s)$ near the line, $\sigma = 1$, we use a particular representation of $\zeta(s)$ obtained from the following result.

Theorem 6.2.1.: *For any integer N ≥ 0 and σ > 0,*

$$\zeta(s)\sum_{n=0}^{N}\frac{1}{n^s}+\frac{N^{1-s}}{s-1}-s\int_{N}^{\infty}\frac{x-[x]}{x^{s+1}}dx \qquad\qquad ...(6.8)$$

Proof: Apply Euler's Summation Formula with $f(t)=\dfrac{1}{t^s}$ and with integers *x* and *y* to obtain,

$$\sum_{y<n\leq x}\frac{1}{n^s}=\int_{y}^{x}\frac{dt}{t^s}-s\int_{y}^{x}\frac{t-[t]}{t^{s+1}}dt$$

Taking *y* = *N* and, letting $x \to \infty$, keeping $\sigma > 1$, this yields,

$$\sum_{n=N+1}^{\infty}\frac{1}{n^s}=\int_{N}^{\infty}\frac{dt}{t^s}-s\int_{N}^{\infty}\frac{t-[t]}{t^{s+1}}dt$$

or,

$$\zeta(s)-\sum_{n=0}^{N}\frac{1}{n^s}=\frac{N^{1-s}}{s-1}-s\int_{N}^{\infty}\frac{x-[x]}{t^{x+1}}dx$$

This proves (6.8) for $\sigma > 1$. For the case when $0 < \delta \leq \sigma$, the integral is dominated by, $\displaystyle\int_{N}^{\infty}\frac{1}{t^{\delta+1}}dx$, so it converges

uniformly for $\sigma \geq \delta$ and hence represents an analytic function in the half-plane $\sigma > 0$. Therefore, (6.8) holds for $\sigma > 0$ by analytic continuation.

Remark 6.2.2.: We also deduce the formula of $\zeta'(s)$ obtained by differentiating each member of (6.8),

$$\zeta'(s)=-\sum_{n=1}^{N}\frac{\log n}{n^s}+s\int_{N}^{\infty}\frac{(x-[x])\log x}{x^{s+1}}dx-\int_{N}^{\infty}\frac{(x-[x])}{x^{s+1}}dx$$

$$-\frac{N^{1-s}\log N}{s-1}-\frac{N^{1-s}}{(s-1)^2} \qquad\qquad ...(6.9)$$

The next theorem uses the relations derived to obtain an upper bound for both $|\zeta(s)|$ and $|\zeta'(s)|$.

Theorem 6.2.3.: *For every A > 0, $\exists$ a constant M (depending on A) such that,*

$$|\zeta(s)| \leq M \log t \text{ and } |\zeta'(s)| \leq M \log^2 t \qquad \qquad \text{...(6.10)}$$

$\forall$ s with $\sigma \geq \dfrac{1}{2}$ satisfying,

$$\sigma > 1 - \frac{A}{\log t} \text{ and } t \geq e \qquad \qquad \text{...(6.11)}$$

Proof: If $\sigma \geq 2$, we have, $|\zeta(s)| \leq \zeta(2)$ and $|\zeta'(s)| \leq |\zeta'(2)|$. Hence, the inequalities in (6.10) are trivially satisfied.

Therefore, we can assume, $\sigma < 2$ and $t \geq e$. Consequently, we get,

$|s| \leq \sigma + t \leq 2 + t < 2t$ and $|s - 1| \geq t$.

Hence, $\dfrac{1}{|s-1|} \leq \dfrac{1}{t}$. Estimating $|\zeta(s)|$ by using (6.8) yields,

$$|\zeta(s)| \leq \sum_{n=1}^{N} \frac{1}{n^\sigma} + 2t \int_N^\infty \frac{1}{x^{\sigma+1}}\, dx + \frac{N^{1-\sigma}}{t} = \sum_{n=1}^{N} \frac{1}{n^\sigma} + \frac{2t}{\sigma N^\alpha} + \frac{N^{1-\sigma}}{t}$$

A priori we force N to depend on t by choosing, $N = [t]$. Thus, $N \leq t < N + 1$ and $\log n \leq \log t$ if $n \leq N$.

Observe that, (6.11) implies, $1 - \sigma < \dfrac{A}{\log t}$. Consequently,

$$\frac{1}{n^\sigma} = \frac{n^{1-\sigma}}{n} = \frac{1}{n} e^{(1-\sigma)\log n} < \frac{1}{n} e^{A\frac{\log n}{\log t}} \leq \frac{1}{n} e^A = O\left(\frac{1}{n}\right)$$

Therefore,

$$\frac{2t}{\sigma N^\sigma} \leq \frac{N+1}{N} \text{ and } \frac{N^{1-\sigma}}{t} = \frac{N}{t} \cdot \frac{1}{N^\sigma} = O\left(\frac{1}{N}\right) = O(1)$$

So,

$$|\zeta(s)| = O\left(\sum_{n=1}^{N} \frac{1}{n}\right) + O(1) = O(\log(N)) + O(1) = O(\log(t))$$

This proves the inequality for $|\zeta(s)|$ in (6.10).

To obtain the expression for $|\zeta'(s)|$ we apply similar arguments on the Equation (6.9). It is significant to note that, an extra factor log$N$ appears on R.H.S. Although the following estimate, log$N = O(\log t)$ gives, $|\zeta'(s)| = O(\log^2 t)$ in the specified region.

6.3. The Non-Vanishing of ζ(s) on the line, σ = 1

As evident from the title, in this section, we indeed wish to establish that, $\zeta(1 + it) \neq 0, \ \forall t \in \mathbb{R}$. The proof primarily focuses on an inequality.

Theorem 6.3.1.: *Given that the Dirichlet Series* $F(s) = \sum \dfrac{f(n)}{n^s}$ *corresponding to an arithmetic function f(n) be absolutely convergent for* $\sigma > \sigma_a$. *We further assume, f(1) ≠ 0. If F(s) ≠ 0 for* $\sigma > \sigma_0 > \sigma_a$, *then, for* $\sigma > \sigma_0$,

$$G(s) = \log f(1) + \sum_{n=2}^{\infty} \frac{(f' * f^{-1})(n)}{\log n} \frac{1}{n^s}$$

where, f^{-1} is the Dirichlet Inverse of n and, f' (n) = f(n) logn.

Remark 6.3.2.: For complex $z \neq 0$, log z denotes that branch of the logarithm which is real when $z > 0$.

Proof: Since, $F(s) \neq 0$, we can write, $F(s) = e^{G(s)}$ for some function $G(s)$, which is analytic for $\sigma > \sigma_0$. Differentiation gives us,

$$F'(s) = e^{G(s)}G'(s) = F(s)G'(s)$$

Therefore, we get, $G'(s) = \dfrac{F'(s)}{F(s)}$. Although, we have,

$$F'(s) = -\sum_{n=1}^{\infty} \frac{f(n)\log n}{n^s} = -\sum_{n=1}^{\infty} \frac{f'(n)}{n^s} \quad \text{and} \quad \frac{1}{F(s)} = \sum_{n=1}^{\infty} \frac{f^{-1}(n)}{n^s}$$

Hence, $G'(s) = \dfrac{F'(s)}{F(s)} = -\sum_{n=2}^{\infty} \dfrac{(f' * f^{-1})(n)}{n^s}$

Integration gives,

$$G(s) = C + \sum_{n=2}^{\infty} \frac{(f' * f^{-1})(n)}{n^s \log n}$$

Where, C is a constant. As, $\sigma \to +\infty$, we obtain,

$$\lim_{\sigma \to +\infty} G(\sigma + it) = C$$

Therefore,

$$f(1) = \lim_{\sigma \to +\infty} F(\sigma + it) = e^C$$

Thus, $C = \log f(1)$ and this completes the proof.

Remark 6.3.3.: It can also be deduced from the proof that, the series for $G(s)$ converges absolutely if $\sigma > \sigma_0$.

Corollary 6.3.4.: For the Riemann Zeta Function $\zeta(s)$, we have,

$$\zeta(s) = e^{G(s)}, \text{ for } \sigma > 1$$

where,

$$G(s) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n} n^{-s}$$

Proof: Using $f(n) = 1$, and applying Theorem (6.3.1), we obtain, $f'(n) = \log n$ and $f^{-1}(n) = \mu(n)$,

Therefore,

$$\left(f' * f^{-1}\right)(n) = \sum_{d|n} \log d\, \mu\left(\frac{n}{d}\right) = \Lambda(n)$$

Hence, for $\sigma > 1$, we obtain the result.

As a direct application of Corollary (6.3.4), we prove our main result in this section.

Theorem 6.3.5.: *For $\sigma > 1$,*

$$\zeta^3(\sigma)\left|\zeta(\sigma+it)\right|^4\left|\zeta(\sigma+2it)\right| \geq 1 \qquad\qquad\qquad ...(6.12)$$

Proof: A priori from Corollary (6.3.4), we have, for $\sigma > 1$,

$$\zeta(s) = e^{G(s)}$$

where,

$$G(s) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log(n)\, n^s} = \sum_{p}\sum_{m=1}^{\infty} \frac{1}{mp^{ms}}$$

In other words,

$$\zeta(s) = e^{\left\{\sum_{p}\sum_{m=1}^{\infty} \frac{1}{mp^{ms}}\right\}} = e^{\left\{\sum_{p}\sum_{m=1}^{\infty} \frac{e^{-imt\log(p)}}{mp^{m\sigma}}\right\}}$$

from which, we obtain,

$$\left|\zeta(s)\right| = e^{\left\{\sum_{p}\sum_{m=1}^{\infty} \frac{\cos(mt\log(p))}{mp^{m\sigma}}\right\}}$$

Successive application of this formula with $s = \sigma$, $s = \sigma + it$, $s = \sigma + 2it$ implies,

$$\zeta^3(\sigma)\left|\zeta(\sigma+it)\right|^4\left|\zeta(\sigma+2it)\right| = e^{\left\{\sum_{p}\sum_{m=1}^{\infty} \frac{3+4\cos(mt\log(p))+\cos(2mt\log(p))}{mp^{m\sigma}}\right\}}$$

Now the following trigonometric identity,

$$3 + 4\cos\theta + \cos2\theta = 3 + 4\cos\theta + 4\cos^2 2\theta - 1 = 2(1 + \cos\theta)^2 \geq 0.$$

helps us assert that, each term in the last infinite series is non-negative. Consequently, we obtain (6.12).

We thus conclude the following.

Theorem 6.3.6.: $\zeta(1+it) \neq 0$ *for every real t.*

Proof: It is sufficient to consider the case when $t \neq 0$. A priori from (6.12) we deduce,

$$(\sigma-1)^3\, \zeta(\sigma)^3 \left|\frac{\zeta(\sigma+it)}{\sigma-1}\right|^4 \left|\zeta(\sigma+2it)\right| \geq \frac{1}{\sigma-1} \qquad\qquad ...(6.13)$$

It is easy to verify that, the above identity is valid for $\sigma > 1$. Now, as $\sigma \to 1^+$ in (6.13), we observe that, the first factor approaches 1 [Since, $\zeta(s)$ has residue 1 at the pole $s = 1$]. The third factor tends to $|\zeta(1 + 2it)|$. If $|\zeta(1 + it)| = 0$, then we could have written the middle factor as,

$$\left| \frac{\zeta(\sigma + it) - \zeta(1 + it)}{\sigma - 1} \right|^4 \to |\zeta(1 + it)|^4 \text{ as } \sigma \to 1^+$$

Thus, for some, $t \neq 0$, if we had, $\zeta(1 + it) = 0$, the L.H.S. of the Equation (6.13) would approach the limit,

$$\left| \zeta'(1 + it) \right|^4 \left| \zeta(1 + 2it) \right| \text{ as } \sigma \to 1^+$$

But, the R.H.S. tends to ∞ as $\sigma \to 1^+$ and this yields a contradiction.

6.4. Inequalities for $\left| \dfrac{1}{\zeta(s)} \right|$ and $\left| \dfrac{\zeta'(s)}{\zeta(s)} \right|$

Applying Theorem (6.3.5), we in fact comment on the bounds of the functions $\left| \dfrac{1}{\zeta(s)} \right|$ and $\left| \dfrac{\zeta'(s)}{\zeta(s)} \right|$.

Theorem 6.4.1.: $\exists$ *a constant $M > 0$ such that,*

$$\left| \frac{1}{\zeta(s)} \right| < M \log^7 t \text{ and } \left| \frac{\zeta'(s)}{\zeta(s)} \right| < M \log^9 t$$

whenever, $\sigma \geq 1$ and, $t \geq e$.

Proof: For $\sigma \geq 2$, we have,

$$\left| \frac{1}{\zeta(s)} \right| = \left| \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \right| \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \leq \zeta(2) \text{ and } \left| \frac{\zeta'(s)}{\zeta(s)} \right| \leq \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^2}$$

Therefore, the inequalities hold trivially provided, $\sigma \geq 2$. Suppose, $1 \leq \sigma \leq 2$ and $t \geq e$. Rewriting inequality (6.12), we get,

$$\frac{1}{|\zeta(\sigma + it)|} \leq (\zeta(\sigma))^{\frac{3}{4}} |\zeta(\sigma + 2it)|^{\frac{1}{4}}$$

Now, observe that, $(\sigma - 1)\zeta(\sigma)$ is bounded in the interval $1 \leq \sigma \leq 2$, say, $(\sigma - 1)\zeta(\sigma) \leq M$, where, M is an absolute constant. Then,

$$\zeta(\sigma) \leq \frac{M}{\sigma - 1} \text{ if } 1 < \sigma \leq 2$$

Furthermore, $\zeta(\sigma + 2it) = O(\log t)$ if, $1 \leq \sigma \leq 2$ [By Theorem (6.2.3)]. Hence, for $1 < \sigma \leq 2$, we have,

$$\frac{1}{|\zeta(\sigma + it)|} \leq \frac{M^{\frac{3}{4}} (\log t)^{\frac{1}{4}}}{(\sigma - 1)^{\frac{3}{4}}} = \frac{A (\log t)^{\frac{1}{4}}}{(\sigma - 1)^{\frac{3}{4}}}$$

where, A is an absolute constant. Therefore, for some constant $B > 0$,

$$\left|\zeta\left(\sigma+it\right)\right| > \frac{B\left(\sigma-1\right)^{\frac{3}{4}}}{\left(\log t\right)^{\frac{1}{4}}} \text{ if } 1 < \sigma \le 2 \text{ and } t \ge e \qquad \qquad \text{...(6.14)}$$

This also holds trivially for $\sigma = 1$. Let, α be any number satisfying, $1 < \sigma < 2$. Then, if $1 \le \sigma \le \alpha$, $t \ge e$, we may use Theorem (6.2.3) to write,

$$\left|\zeta\left(\sigma+it\right) - \zeta\left(\sigma+it\right)\right| \le \int_{\sigma}^{\alpha}\left|\zeta'\left(u+it\right)\right|du \prec \left(\alpha-\sigma\right)M\log^2 t$$

Hence, by triangle inequality,

$$\left|\zeta\left(\sigma+it\right)\right| \ge \left|\zeta\left(\alpha+it\right)\right| - \left|\zeta\left(\sigma+it\right) - \zeta\left(\alpha+it\right)\right| \ge \left|\zeta\left(\alpha+it\right)\right| - \left(\alpha-1\right)M\log^2 t \ge \frac{B\left(\sigma-1\right)^{\frac{3}{4}}}{\left(\log t\right)^{\frac{1}{4}}} - \left(\alpha-1\right)M\log^2 t$$

provided, $1 \le \sigma \le \alpha$, and using (6.14), it also holds for $\alpha \le \sigma \le 2$, since, $\left(\alpha-1\right)^{\frac{3}{4}} \ge \left(\alpha-1\right)^{\frac{3}{4}}$. In other words, $1 \le \sigma \le 2$ and, $t \ge e$, we have the inequality,

$$\left|\zeta\left(\sigma+it\right)\right| \ge \frac{B\left(\sigma-1\right)^{\frac{3}{4}}}{\left(\log t\right)^{\frac{1}{4}}} - \left(\alpha-1\right)M\log^2 t$$

for any α satisfying $1 < \alpha < 2$. Now, we make α depend on t and choose α so the first term on R.H.S. is twice the second. This requires,

$$\alpha = 1 + \left(\frac{B}{2M}\right)^4 \frac{1}{\left(\log t\right)^9}$$

Clearly, $\alpha > 1$ and also $\alpha < 2$ if, $t \ge t_0$ for some t_0. Thus, if, $t \ge t_0$ and $1 \le \sigma \le 2$, we then have,

$$\left|\zeta\left(\sigma+it\right)\right| \ge \left(\alpha-1\right)M\log^2 t = \frac{C}{\left(\log t\right)^7}$$

This inequality should also be true perhaps for a different value of C if, $e \le t \le t_0$.

Hence,

$$\left|\zeta\left(s\right)\right| \ge \frac{C}{\left(\log t\right)^7} \quad \forall\, \sigma \ge 1 \text{ and } t \ge e$$

which gives us a corresponding upper bound for $\left|\dfrac{1}{\zeta\left(s\right)}\right|$

To get the inequality for $\left|\dfrac{\zeta'\left(s\right)}{\zeta\left(s\right)}\right|$, we apply Theorem (6.2.3) to obtain an extra factor $\log^2 t$.

7. Analytic Proof of PNT

Before giving our readers the detailed proof of the "Prime Number Theorem", we shall introduce the following result from complex function theory and another theorem which will be required later on in the proof of the main theorem.

Lemma 7.0.1.: If $f(s)$ has a pole of order k at $s = \alpha$, then the quotient, $\dfrac{\zeta'(s)}{\zeta(s)}$ has a first order pole at $s = \alpha$ with residue $-k$.

Proof: We have, $f(s) = \dfrac{g(s)}{(s-\alpha)^k}$, where, g is analytic at α and $g(\alpha) \neq 0$. Hence, for all s in a neighbourhood of α,

$$f'(s) = \frac{g'(s)}{(s-\alpha)^k} - \frac{kg(s)}{(s-\alpha)^{k+1}} = \frac{g(s)}{(s-\alpha)^k}\left\{\frac{-k}{(s-\alpha)} + \frac{g'(s)}{g(s)}\right\}$$

Consequently,

$$\frac{f'(s)}{f(s)} = \left\{\frac{-k}{(s-\alpha)} + \frac{g'(s)}{g(s)}\right\}$$

Which proves the lemma, since $\dfrac{g'(s)}{g(s)}$ is analytic at α.

Theorem 7.0.2.: *The function,*

$$F(s) = -\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1}$$

is analytic at $s = 1$.

Proof: A priori it follows from the Lemma (7.0.1), $-\dfrac{\zeta'(s)}{\zeta(s)}$ has a first order pole at 1 with residue 1, as does $\dfrac{1}{s-1}$. Hence their difference is analytic at $s = 1$, and the theorem is proved.

Having proven all the necessary results, we can indeed reformulate the statement we intend to establish in order to prove the Prime Number Theorem for our convenience.

Theorem 7.0.3.: *For, $x \geq 1$, we have,*

$$\frac{\psi_1(x)}{x^2} - \frac{1}{2}\left(1 - \frac{1}{x}\right)^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(1+it)e^{it\cdot\log x}dt$$

where, the integral, $\displaystyle\int_{-\infty}^{\infty}\left|h(1+it)\right|dt$ converges.

Therefore, the Riemann-Lebesgue Lemma (*c.f.* Theorem (3.0.1)) implies,

$$\psi_1(x) \sim \frac{x^2}{2} \qquad\qquad\qquad ...(7.1)$$

and hence,

$$\psi(x) \sim x \text{ as, } x \to \infty.$$

which is equivalent to the original statement of "Prime Number Theorem" (2.2.3).

Proof: In Theorem (6.1.2), its already proved that,

$$\frac{\psi_1(x)}{x^2} - \frac{1}{2}\left(1 - \frac{1}{x}\right)^2 = \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} x^{s-1}h(s)\,ds$$

for $c > 1$ and $x \geq 1$, where,

$$h(s) = \frac{1}{s(s+1)}\left\{-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1}\right\}$$

Our first objective is to show that, we can shift the path of integration to the line, $\sigma = 1$. To do this, we apply Cauchy's Theorem (*c.f.* Theorem (3.0.2)) to the rectangle R shown in the Figure 1.

The integral of $x^{s-1}h(s)$ around R is 0, since the integrand is analytic inside and on R. Next, we establish that, the integrals along the horizontal segments tends to 0 as $T \to \infty$. Since the integrand has the same absolute values at the conjugate points, it suffices to consider only the upper segment, $t = T$. On this segment, we have, the estimates,

$$\left|\frac{1}{s(s+1)}\right| \leq \frac{1}{T^2} \text{ and } \left|\frac{1}{s(s+1)(s-1)}\right| \leq \frac{1}{T^3} \leq \frac{1}{T^2}$$
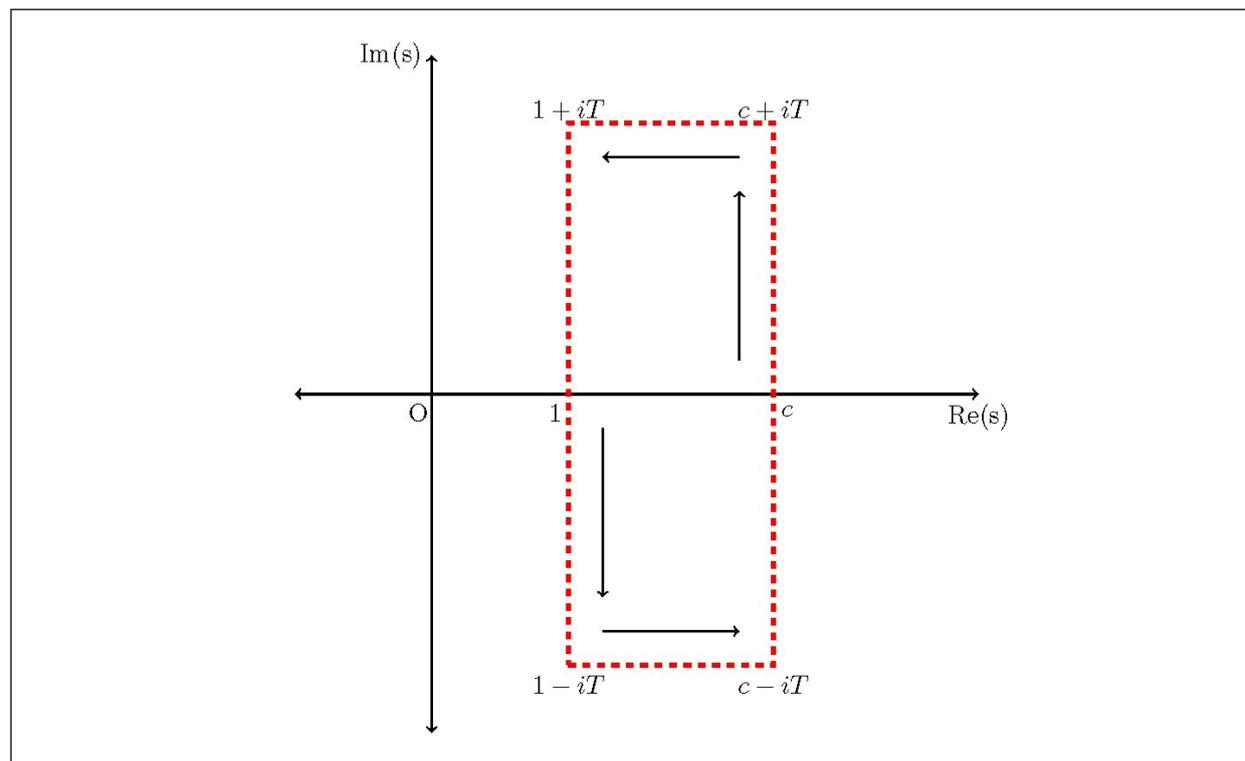

Figure 1

Therefore, $\exists$ a constant M such that, $\left|\dfrac{\zeta'(s)}{\zeta(s)}\right| \le M\left(\log T\right)^9$ if, $\sigma \ge 1$ and $t \ge e$. Hence, if $T \ge e$, then, we shall have,

$$\left|h(s)\right| \le \frac{M\left(\log T\right)^9}{T^2}$$

so that,

$$\left|\int_{1+iT}^{c+iT} x^{s-1}h(s)\,ds\right| \le \int_1^c x^{c-1}\frac{M\left(\log T\right)^9}{T^2}\,d\sigma = Mx^{c-1}\frac{\left(\log T\right)^9}{T^2}(c-1)$$

Similar approach follows for the lower segment, $t = -T$ in order to obtain the corresponding estimate,

$$\left|\int_{1-iT}^{c-iT} x^{s-1}h(s)\,ds\right| \le \int_1^c x^{c-1}\frac{M\left(\log T\right)^9}{T^2}\,d\sigma = Mx^{c-1}\frac{\left(\log T\right)^9}{T^2}(c-1)$$

Therefore, the integrals along the horizontal segments tend to 0 as $T \to \infty$, and,

$$\int_{c-\infty i}^{c+\infty i} x^{s-1}h(s)\,ds = \int_{1-\infty i}^{1+\infty i} x^{s-1}h(s)\,ds$$

On the line $\sigma = 1$, we put $s = 1 + it$ to get,

$$\frac{1}{2\pi i}\int_{1-\infty i}^{1+\infty i} x^{s-1}h(s)\,ds = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(1+it)e^{it\log x}\,dt$$

Now, in the integral, $\displaystyle\int_e^{\infty}\left|h(1+it)\right|dt$, we observe that,

$$\left|h(1+it)\right| \le \frac{M\left(\log t\right)^9}{t^2}$$

Hence, $\displaystyle\int_e^{\infty}\left|h(1+it)\right|dt$ converges. Similarly, $\displaystyle\int_{-\infty}^{-e}\left|h(1+it)\right|dt$ converges, therefore, $\displaystyle\int_{-\infty}^{\infty}\left|h(1+it)\right|dt$ converges.

Thus, we may apply the Riemann-Lebesgue Lemma (*c.f.* Theorem (3.0.1)) to obtain,

$$\psi_1(x) \sim \frac{x^2}{2}$$

By Theorem (4.0.4), the above result implies that,

$$\psi(x) \sim x \ as \ x \to \infty$$

This proves the Prime Number Theorem.

Conflicts of Interest

I as the author of this article declare no conflicts of interest.

Data Availability Statement

I as the sole author of this article confirm that the data supporting the findings of this study are available within the article [and/or] its supplementary materials.

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