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On Nine Duality Principles and Related Convex Dual Formulations Through a D.C. Approach for Non-Convex Optimization

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Abstract

This article develops duality principles and respective convex dual formulations through a D.C. approach applicable to some originally non-convex primal variational formulations. More specifically, in a first step, we develop applications to a Ginzburg-Landau type equation. The results are obtained through basic tools of functional analysis, calculus of variations, duality and optimization theory in infinite dimensional spaces. It is worth emphasizing we have obtained a convex dual variational formulation suitable for a large class of similar models in the calculus of variations.

Keywords: *Duality principle, Ginzburg-Landau system in superconductivity, D.C. approach, Convex dual formulation, Calculus of variations*

1. Introduction

This article develops a duality principle applicable to a large class of models in the calculus of variations. We present applications to a Ginzburg-Landau type equation through a D.C. approach. We recall the so-called D.C. approach refers to a difference between two convex functionals.

More specifically, we obtain a convex dual formulation suitable for an appropriate optimization of a concerning primal functional.

Remark 1.1.: *In particular in this version we present again some important new corrections on Section 10.*

It is worth mentioning the results on duality theory here addressed and developed are inspired mainly in the approaches of J.J.Telega, W.R. Bielski and co-workers presented in the articles (Bielski *et al.*, 1988; Bielski and Telega, 1985; Telega, 1989; Galka and Telega, 1995). Other main reference is the D.C. approach found in the article by Toland (1979).

Moreover, details on the Sobolev spaces involved may be found in Adams and Fournier (2003) and basic theoretical results in superconductivity may be found in Annet (2010) Landau and Lifschits (2008).

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Similar results and models are addressed in Botelho (2021), Botelho (2014), Botelho (2020), Botelho (2009), Botelho (2011) and Botelho (2012).

Basic results on convex analysis are addressed in Rockafellar (1970) and Ekeland and Temam (1976). Finally, other related results may be found in Botelho (2023) and Attouch *et al.* (2006).

Now we start to describe the primal variational formulation for the Ginzburg-Landau model in superconductivity in question.

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded and connected set with a regular (Lipschitzian) boundary by $\partial\Omega$.

Define the Ginzburg-Landau type functional $J : V \to \mathbb{R}$, by

$$J(u) = \frac{\gamma}{2}\int_\Omega \nabla u \cdot \nabla u \; dx$$

$$+\frac{\alpha}{2}\int_\Omega (u^2-\beta)^2 \, dx - \langle u, f\rangle_{L^2} \quad \text{...(1)}$$

Here,

$V = W_0^{1,2}(\Omega)$,

$\gamma > 0$, $\alpha > 0$, $\beta > 0$ and $f \in L^\infty(\Omega)$. We also denote

$Y = Y^* = L^2(\Omega)$.

2. The Main Duality Principle and Related Dual Variational Formulation

In this section we develop in details the main duality principle and respective convex dual variational formulation for the model in question. We highlight some similar results have been obtained in the preprint (Botelho, 2023).

Remark 2.1.: *For $v^* \in L^2(\Omega)$, throughout this text generically we may denote*

$$\int_\Omega \left[\left(-\gamma\nabla^2 + KI_d\right)^{-1} v^*\right](v^*)\,dx$$

simply by

$$\int_\Omega \frac{(v^*)^2}{-\gamma\nabla^2+K}\,dx$$

where I_d denotes the identity operator, $K > 0$, $\gamma > 0$ are real contants and ∇^2 denotes the Laplace operator.

As their meaning are clear, other similar notations may be utilized.

Fix $K_3 = 5$ and $K > 0$ such that

$$K \gg \max\left\{K_3, \gamma, \alpha, \beta, 1/\gamma, 1/\alpha, \beta, \|f\|_\infty\right\}$$

Moreover, define the functionals $F_1 : V \to \mathbb{R}$ and $F_2 : V \times Y^* \to \mathbb{R}$ by

$$F_1(u) = \frac{\gamma}{2}\int_\Omega \nabla u \cdot \nabla u \; dx + \frac{K}{2}\int_\Omega u^2 dx - \langle u, f\rangle_{L^2} \quad \text{...(2)}$$

and

$$F_2\left(u,v_0^*\right)=-\left\langle u^2,v_0^*\right\rangle_{L^2}+\frac{K}{2}\int_\Omega u^2dx$$

$$+\frac{1}{2\alpha}\int_\Omega\left(v_0^*\right)^2dx+\beta\int_\Omega v_0^*\,dx \quad ...(3)$$

Define also,

$$V_1=\left\{u\in V:\|u\|_\infty\le K_3\right\}$$

$$B^*=\left\{v_0^*\in Y^*:\|2v_0^*\|_\infty\le\sqrt{K}/8\right\}$$

and

$$D^*=\left\{v_1^*\in Y^*:\|v_1^*\|_\infty\le(3/2)KK_3\right\}$$

Also, we define the polar functionals $F_1^*:\left[Y^*\right]\to\mathbb{R}$ and $F_2^*:\left[Y^*\right]^2\to\mathbb{R}$ by

$$F_1^*\left(v_1^*\right)=\sup_{u\in V}\left\{\left\langle u,v_1^*\right\rangle_{L^2}-F_1(u)\right\}$$

$$=\frac{1}{2}\int_\Omega\frac{\left(v_1^*+f\right)^2}{-\gamma\nabla^2+K}dx \quad ...(4)$$

and

$$F_2^*\left(v_1^*,v_0^*\right)=\sup_{u\in V}\left\{\left\langle u,v_1^*\right\rangle_{L^2}-F_2\left(u,v_0^*\right)\right\}$$

$$=\frac{1}{2}\int_\Omega\frac{\left(v_1^*\right)^2}{-2v_0^*+K}dx$$

$$-\frac{1}{2\alpha}\int_\Omega\left(v_0^*\right)^2dx-\beta\int_\Omega v_0^*\,dx \quad ...(5)$$

if $v_0^*\in B^*$

Furthermore, define the functional $J^*:D^*\times B^*\to\mathbb{R}$ by

$$J^*\left(v_1^*,v_0^*\right)=-F_1^*\left(v_1^*\right)+F_2^*\left(v_1^*,v_0^*\right)$$

and the exactly penalized functional $J_1^*:D^*\times B^*\to\mathbb{R}$ by

$$J_1^*\left(v_1^*,v_0^*\right)=J^*\left(v_1^*,v_0^*\right)$$

$$-\frac{100K^2}{2}\left\|-\left(\frac{v_1^*+f}{-\gamma\nabla^2+K}\right)+\frac{v_1^*}{-2v_0^*+K}\right\|_{0,2}^2 \quad ...(6)$$

Let $\left(\hat{v}_1^*, \hat{v}_0^*\right) \in D^* \times B^*$ be such that

$$\delta J^*\left(\hat{v}_1^*, \hat{v}_0^*\right) = 0$$

From this and from the Legendre transform proprieties, for

$$u_0 = \frac{\hat{v}_1^*}{-2\hat{v}_0^* + K} \in V$$

we obtain

$$\delta J(u_0) = 0,$$

$$J\left(u_0\right) = J^*\left(\hat{v}_1^*, \hat{v}_0^*\right) = J_1^*\left(\hat{v}_1^*, \hat{v}_0^*\right)$$

and

$$\delta J_1^*\left(\hat{v}_1^*, \hat{v}_0^*\right) = 0$$

Observe that

$$\frac{\partial^2 J_1^*\left(v_1^*, v_0^*\right)}{\partial\left(v_1^*\right)^2}$$

$$= -\frac{1}{-\gamma\nabla^2 + K} + \frac{1}{-2\hat{v}_0^* + K}$$

$$-100K^2\left(-\frac{1}{-\gamma\nabla^2 + K} + \frac{1}{-2v_0^* + K}\right)^2$$

$$< 0 \qquad \text{...(7)}$$

Moreover,

$$\frac{\partial^2 J_1^*\left(v_1^*, v_0^*\right)}{\partial\left(v_0^*\right)^2}$$

$$= -\frac{1}{\alpha} + \frac{4u^2}{-2v_0^* + K}$$

$$-100K^2\left(\frac{4u^2}{\left(-2v_0^* + K\right)^2}\right) + 100K^2\mathcal{O}\left(\frac{1}{K^3}\right)$$

$$< 0 \qquad \text{...(8)}$$

and

$$\frac{\partial^2 J_1^*\left(v_1^*, v_0^*\right)}{\partial v_1^* \partial v_0^*}$$

$$= -100K^2\left(\frac{2u}{-2v_0^*+K}\right)\left(-\frac{1}{-\gamma\nabla^2+K}+\frac{1}{-2v_0^*+K}\right)+\frac{2u}{-2v_0^*+K}$$

$$+100K^2\mathcal{O}\left(\frac{1}{K^3}\right) \quad ...(9)$$

in $D^* \times B^*$.

Thus, by direct computation, we may obtain

$$\det\left\{\frac{\partial^2 J_1^*(v_1^*,v_0^*)}{\partial v_0^*\partial v_1^*}\right\}=\left(\frac{\partial^2 J_1^*(v_1^*,v_0^*)}{\partial (v_1^*)^2}\right)\left(\frac{\partial^2 J_1^*(v_1^*,v_0^*)}{\partial (v_0^*)^2}\right)-\left(\frac{\partial^2 J_1^*(v_1^*,v_0^*)}{\partial v_1^*\partial v_0^*}\right)^2>0$$

in $D^* \times B^*$.

From such results, we may infer that J_1^* is concave in $D^* \times B^*$, so that

$$J_1^*(\hat{v}_1^*\hat{v}_0^*)=\sup_{(v_1^*,v_0^*)\in D^*\times B^*} J_1^*(v_1^*,v_0^*)$$

On the other hand

$$\left(\frac{\partial\left(J(u)+\frac{100K^2}{2}\|u-u_0\|_0^2\right)}{\partial u}\right)\Bigg|_{u=u_0}=\frac{\partial J(u)}{\partial u}\Bigg|_{u=u_0}=0$$

so that by an evident convexity, we have obtained

$$J(u_0)=\inf_{u\in V}\left\{J(u)+\frac{100K^2}{2}\|u-u_0\|_0^2\right\}$$

Joining the pieces, we have got,

$$J(u_0)=\inf_{u\in V_1}\left\{J(u)+\frac{100K^2}{2}\|u-u_0\|_{0,2}^2\right\}$$

$$=J_1^*(\hat{v}_1^*,\hat{v}_0^*)=\sup_{(v_1^*,v_0^*)\in D^*\times B^*} J_1^*(v_1^*,v_0^*) \quad ...(10)$$

The objective of this section is complete.

3. Another Duality Principle

Let $0 < \varepsilon \ll 1$. Define also in this section $K_3 = 3$ and $K = 5$ and assume $\gamma, \alpha, \beta, \|f\|_\infty = \mathcal{O}(1)$.

Define the functionals $F_1 : V \to \mathbb{R}$, $F_2 : V \times Y^* \to \mathbb{R}$ and $F_3 : V \to \mathbb{R}$ by

$$F_1(u) = \frac{\varepsilon}{2}\int_\Omega u^2\,dx - \langle u, f\rangle_{L^2} \quad ...(11)$$

$$F_2(u, v_0^*) = \langle u^2, v_0^*\rangle_{L^2} = \frac{5\gamma}{2}\int_\Omega \nabla u \cdot \nabla u\;dx$$

$$-\frac{\varepsilon}{2}\int_\Omega u^2\,dx - \frac{1}{2\alpha}\int_\Omega (v_0^*)^2\,dx - \beta\int_\Omega v_0^*\,dx \quad ...(12)$$

and

$$F_3(u) = \frac{4\gamma}{2}\int_\Omega \nabla u \cdot \nabla u\;dx$$

Define also,

$$V_1 = \left\{u \in V : \|u\|_\infty \le K_3 \text{ and } uf \ge 0, \text{ in } \Omega\right\}$$

$$B^* = \left\{v_0^* \in V^* : \|2v_0^*\|_\infty \le K\right\}$$

$$D^* = \left\{v_1^* \in Y^* : \|v_1^*\|_\infty \le K\right\}$$

and

$$E^* = \left\{z^* \in Y^* : \|z^*\|_\infty \le K \text{ and } \left[\left(-\gamma\nabla^2\right)^{-1} z^*\right] f \ge 0, \text{ in } \Omega\right\}$$

Moreover, define the polar functionals $F_1^* : [Y^*] \to \mathbb{R}$ and $F_2^* : [Y^*]^2 \to \mathbb{R}$ by

$$F_1^*(v_1^*) = \sup_{u\in V}\left\{\langle u, v_1^*\rangle_{L^2} - F_1(u)\right\}$$

$$= \frac{1}{2}\int_\Omega \frac{(v_1^* + f)^2}{\varepsilon}\,dx \quad ...(13)$$

$$F_2^*(v_1^*, v_0^*) = \sup_{u\in V}\left\{\langle u, -v_1^* + z^*\rangle_{L^2} - F_2(u, v_0^*)\right\}$$

$$= \frac{1}{2}\int_\Omega \frac{(-v_1^* + z^*)^2}{-5\gamma\nabla^2 + 2v_0^* - \varepsilon}\,dx$$

$$= \frac{1}{2\alpha}\int_\Omega (v_0^*)^2\,dx + \beta\int_\Omega v_0^*\,dx \quad ...(14)$$

if $v_0^* \in B^*$ and

$$F_3^*(z^*) = \sup_{u\in V}\left\{\langle u, z^*\rangle_{L^2} - F_3(u)\right\}$$

$$= \frac{1}{8}\int_\Omega \frac{(z^*)^2}{-\gamma\nabla^2}\,dx \qquad ...(15)$$

Furthermore, define the functional $J^* : D^* \times B^* \times E^* \to \mathbb{R}$ by

$$J^*(v_1^*, v_0^*, z^*) = -F_1^*(v_1^*) - F_2^*(v_1^*, z^*, v_0^*) + F_3^*(z^*)$$

and the exactly penalized functional $J_1^* : D^* \times B^* \times E^* \to \mathbb{R}$ by

$$J_1^*(v_1^*, v_0^*, z^*) = J^*(v_1^*, v_0^*, z^*)$$

$$+\frac{1}{6}\left\|\sqrt{-\gamma\nabla^2}\left(\frac{v_1^*}{-\gamma\nabla^2 + 2v_0^* - \varepsilon}\right) + \frac{z^*}{-4\gamma\nabla^2}\right\|_{0,2}^2 \qquad ...(16)$$

Let $(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*) \in D^* \times B^* \times E^*$ be such that

$$\delta J^*(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*) = 0$$

From this and from the Legendre transform proprieties, for

$$u_0 = \frac{\hat{z}^*}{-4\gamma\nabla^2} \in V$$

we obtain

$$\delta J(u_0) = 0,$$

$$J(u_0) = J^*(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*) = J_1^*(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*)$$

and

$$\delta J_1^*(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*) = 0$$

Observe that

$$\frac{\partial^2 J_1^*(v_1^*, v_0^*, z^*)}{\partial (z^*)^2}$$

$$= -\frac{1}{-5\gamma\nabla^2 + 2v_0^* - \varepsilon} + \frac{1}{-4\gamma\nabla^2} + \frac{1}{48(-\gamma\nabla^2)}$$

$$= \frac{-\gamma\nabla^2 + 2v_0^* - \varepsilon}{(-5\gamma\nabla^2 + 2v_0^* - \varepsilon)(-4\gamma\nabla^2)} + \frac{1}{48(-\gamma\nabla^2)}$$

$$= \frac{68(-\gamma\nabla^2) + 52(2v_0^*) - 52\varepsilon}{(-5\gamma\nabla^2 + 2v_0^* - \varepsilon)(-4\gamma\nabla^2)48} \qquad ...(17)$$

Here we assume that

$$67\left(-\gamma\nabla^2\right)+52\left(2\hat{v}_0^*\right)>0$$

so that, since J_1^* is quadratic in z^*, we obtain

$$J_1^*\left(\hat{v}_1^*,\hat{v}_0^*,\hat{z}^*\right)=\inf_{z^*\in E^*}J_1^*\left(\hat{v}_1^*,\hat{v}_0^*,z^*\right)$$

Moreover, we assume γ, α, β are such that

$$\frac{\partial^2 J_1^*\left(v_1^*,v_0^*,\hat{z}^*\right)}{\partial\left(v_0^*\right)^2}$$

$$<-\frac{1}{10}I_d \qquad ...(18)$$

and

$$\frac{\partial^2 J_1^*\left(v_1^*,v_0^*,\hat{z}^*\right)}{\partial v_1^*\partial v_0^*}$$

$$\approx\mathcal{O}(1) \qquad ...(19)$$

in $D^*\times B^*$.

From such assumptions and results, since

$$\frac{\partial^2 J_1^*\left(v_1^*,v_0^*\right)}{\partial\left(v_1^*\right)^2}=-\mathcal{O}\left(1/\varepsilon\right)$$

we have that

$$\det\left\{\frac{\partial^2 J_1^*\left(v_1^*,v_0^*,\hat{z}^*\right)}{\partial v_0^*\partial v_1^*}\right\}=\left(\frac{\partial^2 J_1^*\left(v_1^*,v_0^*,\hat{z}^*\right)}{\partial\left(v_1^*\right)^2}\right)\left(\frac{\partial^2 J_1^*\left(v_1^*,v_0^*,\hat{z}^*\right)}{\partial\left(v_0^*\right)^2}\right)-\left(\frac{\partial^2 J_1^*\left(v_1^*,v_0^*,\hat{z}^*\right)}{\partial v_1^*\partial v_0^*}\right)^2>0$$

in $D^*\times B^*$.

From such results, we may infer that $J_1^*\left(v_1^*,v_0^*,\hat{z}^*\right)$ is concave in $\left(v_1^*,v_0^*\right)$ on $D^*\times B^*$, so that

$$J_1^*\left(\hat{v}_1^*,\hat{v}_0^*,\hat{z}^*\right)=\sup_{\left(v_1^*,v_0^*\right)\in D^*\times B^*}J_1^*\left(v_1^*,v_0^*,\hat{z}^*\right)$$

From these previous results and a standard Saddle Point Theorem, we have got

$$J_1^*\left(\hat{v}_1^*,\hat{v}_0^*,\hat{z}^*\right)=\sup_{\left(v_1^*,v_0^*\right)\in D^*\times B^*}\left\{\inf_{z^*\in E^*}J_1^*\left(v_1^*,v_0^*,z^*\right)\right\}$$

$$= \inf_{z^* \in E^*} \left\{ \sup_{(v_1^*, v_0^*) \in D^* \times B^*} J_1^*\left(v_1^*, v_0^*, z^*\right) \right\} \quad ...(20)$$

Finally, observe that

$$J(u_0) = J_1^*\left(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*\right)$$

$$\leq F_1(u) + F_2\left(u, \hat{v}_0^*\right) - \left\langle u, z^* \right\rangle_{L^2} + F_3^*\left(z^*\right)$$

$$+ \frac{1}{6} \left\| \sqrt{-\gamma \nabla^2} \left(-u_0 + \frac{z^*}{-4\gamma \nabla^2} \right) \right\|_{0,2}^2 \quad ...(21)$$

$\forall u \in V_1, z^* \in E^*$

In particular, for $z^* = -4\gamma \nabla^2 u$, we obtain

$$J(u_0) = J_1^*\left(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*\right)$$

$$\leq F_1(u) + F_2\left(u, \hat{v}_0^*\right) - F_3(u)$$

$$+ \frac{1}{6} \left\| \sqrt{-\gamma \nabla^2} \left(-u_0 + u \right) \right\|_{0,2}^2$$

$$\leq \sup_{v_0^* \in Y^*} \left\{ F_1(u) + F_2\left(u, v_0^*\right) - F_3(u) \right\}$$

$$+ \frac{1}{6} \left\| \sqrt{-\gamma \nabla^2} \left(-u_0 + u \right) \right\|_{0,2}^2$$

$$= J(u) + \frac{1}{6} \left\| \sqrt{-\gamma \nabla^2} \left(-u_0 + u \right) \right\|_{0,2}^2 \quad ...(22)$$

$\forall u \in V_1$

Joining the pieces, we have got

$$J(u_0) = \inf_{u \in V_1} \left\{ J(u) + \frac{1}{6} \left\| \sqrt{-\gamma \nabla^2} \left(-u_0 + u \right) \right\|_{0,2}^2 \right\}$$

$$= J_1^*\left(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*\right)$$

$$= \sup_{(v_1^*, v_0^*) \in D^* \times B^*} \left\{ \inf_{z^* \in E^*} J_1^*\left(v_1^*, v_0^*, z^*\right) \right\}$$

$$= \inf_{z^* \in E^*} \left\{ \sup_{(v_1^*, v_0^*) \in D^* \times B^*} J_1^*\left(v_1^*, v_0^*, z^*\right) \right\} \quad ...(23)$$

The objective of this section is complete.

4. One More Duality Principle

Let $0 < \varepsilon \ll 1$. Define also in this section $K_3 = 3$,

$$K \gg \max\{K_3, \gamma, \alpha, \beta\}$$

where we assume, after multiplication by a suitable constant,

$$\gamma, \alpha, \|f\|_\infty = \mathcal{O}(100)$$

We highlight that multiplying $\gamma, \alpha, \|f\|_\infty$ by a constant does not change any critical point since, up to such a multiplying constant, the Euler Lagrange equations keep the same.

Define the functionals $F_1 : V \to \mathbb{R}$, $F_2 : V \times Y^* \to \mathbb{R}$ and $F_3 : V \to \mathbb{R}$ by

$$F_1(u) = \frac{\varepsilon}{2}\int_\Omega u^2 dx - \langle u, f\rangle_{L^2} \quad ...(24)$$

$$F_2(u, v_0^*) = \langle u^2, v_0^*\rangle_{L^2} + \frac{\gamma}{2}\int_\Omega \nabla u \cdot \nabla u \, dx + \frac{K}{2}\int_\Omega u^2 dx$$

$$-\frac{\varepsilon}{2}\int_\Omega u^2 dx - \frac{1}{2\alpha}\int_\Omega (v_0^*)^2 \, dx - \beta\int_\Omega v_0^* \, dx \quad ...(25)$$

and

$$F_3(u) = \frac{K}{2}\int_\Omega u^2 dx$$

Define also,

$$V_1 = \{u \in V : \|u\|_\infty \le K_3 \text{ and } uf \ge 0, \text{ in } \Omega\}$$

$$B^* = \{v_0^* \in V^* : \|2v_0^*\|_\infty \le \sqrt[4]{K}/10\}$$

$$D^* = \{v_1^* \in Y^* : \|v_1^*\|_\infty \le \sqrt[4]{K}\}$$

and

$$E^* = \{z^* \in Y^* : \|z^*\|_\infty \le KK_3 \text{ and } z^* f \ge 0, \text{ in } \Omega\}$$

Moreover, define the polar functionals $F_1^* : [Y^*] \to \mathbb{R}$ and $F_2^* : [Y^*]^2 \to \mathbb{R}$ by

$$F_1^*(v_1^*) = \sup_{u \in V}\{\langle u, v_1^*\rangle_{L^2} - F_1(u)\}$$

$$= \frac{1}{2}\int_\Omega \frac{(v_1^* + f)^2}{\varepsilon} dx \quad ...(26)$$

$$F_2^*\left(v_1^*, v_0^*\right) = \sup_{u \in V}\left\{\left\langle u, -v_1^* + z^*\right\rangle_{L^2} - F_2\left(u, v_0^*\right)\right\}$$

$$= \frac{1}{2}\int_\Omega \frac{\left(-v_1^* + z^*\right)^2}{-\gamma\nabla^2 + 2v_0^* + K - \varepsilon}\,dx$$

$$= \frac{1}{2\alpha}\int_\Omega \left(v_0^*\right)^2 dx + \beta\int_\Omega v_0^*\,dx \quad \text{...(27)}$$

if $v_0^* \in B^*$. and

$$F_3^*\left(z^*\right) = \sup_{w \in L^2}\left\{\left\langle w, z^*\right\rangle_{L^2} - F_3\left(w\right)\right\}$$

$$= \frac{1}{2K}\int_\Omega \left(z^*\right)^2 dx \quad \text{...(28)}$$

Furthermore, define the functional $J^* : D^* \times B^* \times E^* \to \mathbb{R}$ by

$$J^*\left(v_1^*, v_0^*, z^*\right) = -F_1^*\left(v_1^*\right) - F_2^*\left(v_1^*, z^*, v_0^*\right) + F_3^*\left(z^*\right)$$

and the exactly penalized functional $J_1^* : D^* \times B^* \times E^* \to \mathbb{R}$ by

$$J_1^*\left(v_1^*, v_0^*, z^*\right) = J^*\left(v_1^*, v_0^*, z^*\right)$$

$$+\frac{10}{2}\left\|\frac{v_1^*}{-\gamma\nabla^2 + 2v_0^* - \varepsilon} + \frac{z^*}{K}\right\|_{0,2}^2 \quad \text{...(29)}$$

Let $\left(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*\right) \in D^* \times B^* \times E^*$ be such that

$$\delta J^*\left(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*\right) = 0$$

From this and from the Legendre transform proprieties, for

$$u_0 = \frac{\hat{z}^*}{K} \in V_1$$

we obtain

$$\delta J(u_0) = 0,$$

$$J\left(u_0\right) = J^*\left(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*\right) = J_1^*\left(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*\right)$$

and

$$\delta J_1^*\left(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*\right) = 0$$

Observe that

$$\frac{\partial^2 J_1^*\left(v_1^*, v_0^*, z^*\right)}{\partial\left(z^*\right)^2}$$

$$= -\frac{1}{-\gamma\nabla^2 + 2v_0^* + K - \varepsilon} + \frac{1}{K} + \frac{10}{K^2}$$

$$= \frac{-\gamma\nabla^2 + 2v_0^* - \varepsilon}{\left(-\gamma\nabla^2 + 2v_0^* + K - \varepsilon\right)(K)} + \frac{10}{K^2}$$

$$= \frac{\left(-\gamma\nabla^2 + 2v_0^* - \varepsilon + 10\right)K + 10\left(-\gamma\nabla^2 + 2v_0^* - \varepsilon\right)}{\left(-\gamma\nabla^2 + 2v_0^* + K - \varepsilon\right)10K^2}$$

$$= \mathcal{O}\left(\frac{\left(-\gamma\nabla^2 + 2v_0^* - \varepsilon + 10\right)}{10K^2}\right) + \mathcal{O}\left(1/K^3\right) \qquad \text{...(30)}$$

Here we assume that

$$\left(-\gamma\nabla^2\right) + \left(2\hat{v}_0^*\right) + 9 > 0$$

so that, since J_1^* is quadratic in z^*, we obtain

$$J_1^*\left(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*\right) = \inf_{z^* \in E^*} J_1^*\left(\hat{v}_1^*, \hat{v}_0^*, z^*\right)$$

Moreover, we assume γ, α, β are such that

$$\frac{\partial^2 J_1^*\left(v_1^*, v_0^*, \hat{z}^*\right)}{\partial\left(v_0^*\right)^2}$$

$$= -\frac{1}{\alpha} + \mathcal{O}\left(\frac{10}{100^2}\right)$$

$$< -\frac{1}{2\alpha} \qquad \text{...(31)}$$

and

$$\frac{\partial^2 J_1^*\left(v_1^*, v_0^*, \hat{z}^*\right)}{\partial v_1^* \partial v_0^*}$$

$$\approx \mathcal{O}\left(\frac{10}{100^2}\right) \qquad \text{...(32)}$$

in $D^* \times B^*$.

From such assumptions and results, since

$$\frac{\partial^2 J_1^*\left(v_1^*, v_0^*, \hat{z}^*\right)}{\partial\left(v_1^*\right)^2} = -\mathcal{O}(1/\varepsilon)$$

we have that

$$\det\left\{\frac{\partial^2 J_1^*\left(v_1^*, v_0^*, \hat{z}^*\right)}{\partial v_0^* \partial v_1^*}\right\} = \left(\frac{\partial^2 J_1^*\left(v_1^*, v_0^*, \hat{z}^*\right)}{\partial\left(v_1^*\right)^2}\right)\left(\frac{\partial^2 J_1^*\left(v_1^*, v_0^*, \hat{z}^*\right)}{\partial\left(v_0^*\right)^2}\right) - \left(\frac{\partial^2 J_1^*\left(v_1^*, v_0^*, \hat{z}^*\right)}{\partial v_1^* \partial v_0^*}\right)^2 > 0$$

in $D^* \times B^*$.

From such results, we may infer that $J_1^*\left(v_1^*, v_0^*, \hat{z}^*\right)$ is concave in $\left(v_1^*, v_0^*\right)$ on $D^* \times B^*$, so that

$$J_1^*\left(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*\right) = \sup_{\left(v_1^*, v_0^*\right) \in D^* \times B^*} J_1^*\left(v_1^*, v_0^*, \hat{z}^*\right)$$

From these previous results and a standard Saddle Point Theorem, we have got

$$J_1^*\left(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*\right) = \sup_{\left(v_1^*, v_0^*\right) \in D^* \times B^*} \left\{\inf_{z^* \in E^*} J_1^*\left(v_1^*, v_0^*, z^*\right)\right\}$$

$$= \inf_{z^* \in E^*} \left\{\sup_{\left(v_1^*, v_0^*\right) \in D^* \times B^*} J_1^*\left(v_1^*, v_0^*, z^*\right)\right\} \quad ...(33)$$

Finally, observe that

$$J\left(u_0\right) = J_1^*\left(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*\right)$$

$$\leq F_1(u) + F_2\left(u, \hat{v}_0^*\right) - \left\langle u, z^*\right\rangle_{L^2} + F_3^*\left(z^*\right)$$

$$+\frac{10}{2}\left\|\left(-u_0 + \frac{z^*}{K}\right)\right\|_{0,2}^2 \quad ...(34)$$

$$\forall u \in V_1, z^* \in E^*$$

In particular, for $z^* = Ku$, we obtain

$$J\left(u_0\right) = J_1^*\left(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*\right)$$

$$\leq F_1(u) + F_2\left(u, \hat{v}_0^*\right) - F_3(u)$$

$$+\frac{10}{2}\left\|\left(-u_0 + u\right)\right\|_{0,2}^2$$

$$\leq \sup_{v_0^* \in Y^*} \left\{ F_1(u) + F_2(u, v_0^*) - F_3(u) \right\}$$

$$+ \frac{10}{2} \|(-u_0 + u)\|_{0,2}^2$$

$$= J(u) + \frac{10}{2} \|(-u_0 + u)\|_{0,2}^2 \quad ...(35)$$

$\forall u \in V_1$

Joining the pieces, we have got

$$J(u_0) = \inf_{u \in V_1} \left\{ J(u) + \frac{10}{2} \|(-u_0 + u)\|_{0,2}^2 \right\}$$

$$= J_1^*(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*)$$

$$= \sup_{(v_1^*, v_0^*) \in D^* \times B^*} \left\{ \inf_{z^* \in E^*} J_1^*(v_1^*, v_0^*, z^*) \right\}$$

$$= \inf_{z^* \in E^*} \left\{ \sup_{(v_1^*, v_0^*) \in D^* \times B^*} J_1^*(v_1^*, v_0^*, z^*) \right\} \quad ...(36)$$

The objective of this section is complete.

5. A Fourth Duality Principle

Define $K_3 = 3,\ K \gg \max\{K_3, \gamma, \alpha, \beta\}$ and assume $\gamma, \alpha, \beta, \|f\|_\infty = \mathcal{O}(1)$.

Define the functionals $F_1 : V \times Y^* \to \mathbb{R}$, $F_2 : V \times Y^* \to \mathbb{R}$ and $F_3 : V \to \mathbb{R}$ by

$$F_1(u, v_0^*) = \frac{\gamma}{2} \int_\Omega \nabla u \cdot \nabla u \, dx + \frac{K}{2} \int_\Omega u^2 dx$$

$$+ \langle u^2, v_0^* \rangle_{L^2} - \langle u, f \rangle_{L^2} \quad ...(37)$$

$$F_2(u, v_0^*) = \frac{K}{2} \int_\Omega u^2 dx$$

$$- \frac{1}{2\alpha} \int_\Omega (v_0^*)^2 \, dx - \beta \int_\Omega v_0^* \, dx \quad ...(38)$$

and

$$F_3(u) = \frac{2K}{2} \int_\Omega u^2 dx$$

Define also,

$$V_1 = \left\{ u \in V : \|u\|_\infty \leq K_3 \text{ and } uf \geq 0, \text{ in } \Omega \right\}$$

$$B^* = \left\{ v_0^* \in V^* : \|2v_0^*\|_\infty \leq \sqrt[4]{K}/8 \right\}$$

$$D^* = \left\{ v_1^* \in Y^* : \|v_1^*\|_\infty \leq (3/2)KK_3 \right\}$$

and

$$E^* = \left\{ z^* \in Y^* : \|z^*\|_\infty \leq 2KK_3 \text{ and } z^* f \geq 0, \text{ in } \Omega \right\}$$

Moreover, define the polar functionals $F_1^* : \left[Y^*\right]^2 \to \mathbb{R}$, $F_2^* : \left[Y^*\right]^2 \to \mathbb{R}$ and $F_3^* : Y^* \to \mathbb{R}$ by

$$F_1^*\left(v_1^*, v_0^*\right) = \sup_{u \in V} \left\{ \left\langle u, v_1^* \right\rangle_{L^2} - F_1\left(u, v_0^*\right) \right\}$$

$$= \frac{1}{2} \int_\Omega \frac{\left(v_1^* + f\right)^2}{-\gamma \nabla^2 + 2v_0^* + K} dx \qquad ...(39)$$

$$F_2^*\left(v_1^*, v_0^*, z^*\right) = \sup_{u \in V} \left\{ \left\langle u, -v_1^* + z^* \right\rangle_{L^2} - F_2\left(u, v_0^*\right) \right\}$$

$$= \frac{1}{2} \int_\Omega \frac{\left(-v_1^* + z^*\right)^2}{K} dx$$

$$+ \frac{1}{2\alpha} \int_\Omega \left(v_0^*\right)^2 dx + \beta \int_\Omega v_0^* \, dx \qquad ...(40)$$

if $v_0^* \in B^*$. and

$$F_3^*\left(z^*\right) = \sup_{w \in L^2} \left\{ \left\langle w, z^* \right\rangle_{L^2} - F_3(w) \right\}$$

$$= \frac{1}{4K} \int_\Omega \left(z^*\right)^2 dx \qquad ...(41)$$

Furthermore, define the functional $J^* : B^* \times D^* \times E^* \to \mathbb{R}$ by

$$J^*\left(v_1^*, v_0^*, z^*\right) = -F_1^*\left(v_1^*\right) - F_2^*\left(v_1^*, z^*, v_0^*\right) + F_3^*\left(z^*\right)$$

and the exactly penalized functional $J_1^* : D^* \times B^* \times E^* \to \mathbb{R}$ by

$$J_1^*\left(v_1^*, v_0^*, z^*\right) = J^*\left(v_1^*, v_0^*, z^*\right)$$

$$- \frac{100K^2}{2} \left\| - \frac{v_1^* + f}{-\gamma \nabla^2 + 2v_0^* + K} + \frac{v_1^*}{K} \right\|_{0,2}^2$$

$$+\frac{2K}{2}\left\|-\frac{v_1^*}{K}+\frac{z^*}{2K}\right\|_{0,2}^2 \quad ...(42)$$

Let $\left(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*\right) \in D^* \times B^* \times E^*$ be such that

$$\delta J^*\left(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*\right) = 0$$

From this and from the Legendre transform proprieties, for

$$u_0 = \frac{\hat{z}^*}{2K} \in V_1$$

we obtain

$$\delta J(u_0) = 0,$$

$$J(u_0) = J^*\left(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*\right) = J_1^*\left(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*\right)$$

and

$$\delta J_1^*\left(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*\right) = 0$$

Observe that

$$\frac{\partial^2 J_1^*\left(v_1^*, v_0^*, z^*\right)}{\partial\left(z^*\right)^2}$$

$$= -\frac{1}{K} + \frac{1}{2K}$$

$$+2\left(\frac{1}{2K}\right)^2$$

$$= -\frac{1}{2K} + \frac{1}{2K}$$

$$= 0 \quad ...(43)$$

so that, since J_1^* (up to an approximate convex regularization) is quadratic in z^*, we obtain

$$J_1^*\left(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*\right) = \inf_{z^* \in E^*} J_1^*\left(\hat{v}_1^*, \hat{v}_0^*, z^*\right)$$

Moreover, for $\left(v_1^*, v_0^*\right) \in D^* \times B^*$, denoting

$$u = \frac{\left(v_1^* + f\right)}{-\gamma \nabla^2 + 2v_0^* + K}$$

we have

$$\frac{\partial^2 J_1^*(v_1^*, v_0^*, \hat{z}^*)}{\partial (v_1^*)^2}$$

$$= -\frac{1}{-\gamma \nabla^2 + 2v_0^* + K} - \frac{1}{K} + \frac{2}{K}$$

$$-100K^2 \left(-\frac{1}{-\gamma \nabla^2 + 2v_0^* + K} + \frac{1}{K} \right)^2$$

$$= \frac{-2K + \gamma \nabla^2 - 2v_0^*}{(-\gamma \nabla^2 + 2v_0^* + K)K} + \frac{2}{K}$$

$$-100K^2 \left(-\frac{1}{-\gamma \nabla^2 + 2v_0^* + K} + \frac{1}{K} \right)^2$$

$$= \frac{(-\gamma \nabla^2 + 2v_0^*)}{(-\gamma \nabla^2 + 2v_0^* + K)K}$$

$$-100K^2 \left(-\frac{1}{-\gamma \nabla^2 + 2v_0^* + K} + \frac{1}{K} \right)^2$$

$$< 0 \qquad \text{...(44)}$$

$$\frac{\partial^2 J_1^*(v_1^*, v_0^*, \hat{z}^*)}{\partial (v_0^*)^2}$$

$$= -\frac{1}{\alpha} - \frac{4u^2}{-\gamma \nabla^2 + 2v_0^* + K}$$

$$-100K^2 \frac{4u^2}{(-\gamma \nabla^2 + 2v_0^* + K)^2} - 100K^2 \mathcal{O}(1/K^3)$$

$$< 0 \qquad \text{...(45)}$$

and

$$\frac{\partial^2 J_1^*(v_1^*, v_0^*, \hat{z}^*)}{\partial v_1^* \partial v_0^*}$$

$$= \frac{2u}{-\gamma \nabla^2 + 2v_0^* + K} - 100K^2 \left(-\frac{1}{-\gamma \nabla^2 + 2v_0^* + K} + \frac{1}{K} \right) \frac{2u}{(-\gamma \nabla^2 + 2v_0^* + K)}$$

$$-100K^2 \mathcal{O}(1/K^3) \qquad \text{...(46)}$$

From such assumptions and results, by direct computation, we may obtain

$$\det\left\{\frac{\partial^2 J_1^*\left(v_1^*,v_0^*,\hat{z}^*\right)}{\partial v_0^*\partial v_1^*}\right\}=\left(\frac{\partial^2 J_1^*\left(v_1^*,v_0^*,\hat{z}^*\right)}{\partial\left(v_1^*\right)^2}\right)\left(\frac{\partial^2 J_1^*\left(v_1^*,v_0^*,\hat{z}^*\right)}{\partial\left(v_0^*\right)^2}\right)-\left(\frac{\partial^2 J_1^*\left(v_1^*,v_0^*,\hat{z}^*\right)}{\partial v_1^*\partial v_0^*}\right)^2>0$$

in $D^*\times B^*$.

From such results, we may infer that $J_1^*\left(v_1^*,v_0^*,\hat{z}^*\right)$ is concave in $\left(v_1^*,v_0^*\right)$ on $D^*\times B^*$, so that

$$J_1^*\left(\hat{v}_1^*,\hat{v}_0^*,\hat{z}^*\right)=\sup_{\left(v_1^*,v_0^*\right)\in D^*\times B^*} J_1^*\left(v_1^*,v_0^*,\hat{z}^*\right)$$

From these previous results and a standard Saddle Point Theorem, we have got

$$J_1^*\left(\hat{v}_1^*,\hat{v}_0^*,\hat{z}^*\right)=\sup_{\left(v_1^*,v_0^*\right)\in D^*\times B^*}\left\{\inf_{z^*\in E^*} J_1^*\left(v_1^*,v_0^*,z^*\right)\right\}$$

$$=\inf_{z^*\in E^*}\left\{\sup_{\left(v_1^*,v_0^*\right)\in D^*\times B^*} J_1^*\left(v_1^*,v_0^*,z^*\right)\right\} \quad ...(47)$$

Finally, observe that

$$J\left(u_0\right)=J_1^*\left(\hat{v}_1^*,\hat{v}_0^*,\hat{z}^*\right)$$

$$\leq F_1\left(u,\hat{v}_0^*\right)+F_2\left(u,\hat{v}_0^*\right)-\left\langle u,z^*\right\rangle_{L^2}+F_3^*\left(z^*\right)$$

$$+\frac{2K}{2}\left\|\left(-u_0+\frac{z^*}{2K}\right)\right\|_{0,2}^2 \quad ...(48)$$

$$\forall u\in V_1, z^*\in E^*$$

In particular, for $z^* = 2Ku$, we obtain

$$J\left(u_0\right)=J_1^*\left(\hat{v}_1^*,\hat{v}_0^*,\hat{z}^*\right)$$

$$\leq F_1\left(u,\hat{v}_0^*\right)+F_2\left(u,\hat{v}_0^*\right)-F_3\left(u\right)$$

$$+\frac{2K}{2}\left\|\left(-u_0+u\right)\right\|_{0,2}^2$$

$$\leq \sup_{v_0^*\in Y^*}\left\{F_1\left(u,v_0^*\right)+F_2\left(u,v_0^*\right)-F_3\left(u\right)\right\}$$

$$+\frac{2K}{2}\left\|\left(-u_0+u\right)\right\|_{0,2}^2$$

$$=J\left(u\right)+K\left\|\left(-u_0+u\right)\right\|_{0,2}^2 \quad ...(49)$$

$\forall u \in V_1$

Joining the pieces, we have got

$$J(u_0) = \inf_{u \in V_1} \left\{ J(u) + K \left\| (-u_0 + u) \right\|_{0,2}^2 \right\}$$

$$= J_1^*\left(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*\right)$$

$$= \sup_{(v_1^*, v_0^*) \in D^* \times B^*} \left\{ \inf_{z^* \in E^*} J_1^*\left(v_1^*, v_0^*, z^*\right) \right\}$$

$$= \inf_{z^* \in E^*} \left\{ \sup_{(v_1^*, v_0^*) \in D^* \times B^*} J_1^*\left(v_1^*, v_0^*, z^*\right) \right\} \quad \text{...(50)}$$

The objective of this section is complete.

6. A Fifth Duality Principle

Define $K_3 = 3,\ K \gg \max\left\{K_3, \alpha, \gamma, \beta, 1/\alpha, 1/\gamma.1/\beta\right\}$.

Define the functionals $F_1 : V \to \mathbb{R}$, $F_2 : V \times Y^* \to \mathbb{R}$ and $F_3 : V \to \mathbb{R}$ by

$$F_1(u) = \frac{\gamma}{2}\int_\Omega \nabla u \cdot \nabla u \ dx - \langle u, f \rangle_{L^2} \quad \text{...(51)}$$

$$F_2\left(u, v_0^*\right) = \left\langle u^2, v_0^* \right\rangle_{L^2} + \frac{K}{2}\int_\Omega u^2 dx$$

$$-\frac{1}{2\alpha}\int_\Omega \left(v_0^*\right)^2 dx - \beta \int_\Omega v_0^* \, dx \quad \text{...(52)}$$

$$F_3(u) = \frac{K}{2}\int_\Omega u^2 dx$$

Define also,

$$V_1 = \left\{ u \in V : \|u\|_\infty \le K_3 \text{ and } uf \ge 0, \text{ in } \Omega \right\}$$

$$B^* = \left\{ v_0^* \in Y^* : \left\| 2v_0^* \right\|_\infty \le \sqrt[4]{K} \right\}$$

$$D^* = \left\{ v_1^* \in Y^* : \left\| v_1^* \right\|_\infty \le \sqrt[8]{K} \right\}$$

and

$$E^* = \left\{ z^* \in \left[Y^*\right] : \left\| z^* \right\|_\infty \le KK_3 \text{ and } z^* f \ge 0, \text{ in } \Omega \right\}$$

Moreover, define the polar functionals $F_1^* : \left[Y^*\right]^2 \to \mathbb{R}$, $F_2^* : \left[Y^*\right]^2 \to \mathbb{R}$ and $F_3^* : Y^* \to \mathbb{R}$ by

$$F_1^*(v_1^*) = \sup_{u \in V} \left\{ \langle u, v_1^* \rangle_{L^2} - F_1(u) \right\}$$

$$= \frac{1}{2} \int_\Omega \frac{(v_1^* + f)^2}{-\gamma \nabla^2} \, dx \quad ...(53)$$

$$F_2^*(v_1^*, v_0^*, z^*) = \sup_{u \in V} \left\{ \langle u, -v_1^* + z^* \rangle_{L^2} - F_2(u, v_0^*) \right\}$$

$$= \frac{1}{2} \int_\Omega \frac{(-v_1^* + z^*)^2}{2v_0^* + K} \, dx$$

$$+ \frac{1}{2\alpha} \int_\Omega (v_0^*)^2 \, dx + \beta \int_\Omega v_0^* \, dx \quad ...(54)$$

if $v_0^* \in B^*$ and

$$F_3^*(z^*) = \sup_{w \in L^2} \left\{ \langle w, z^* \rangle_{L^2} - F_3(w) \right\}$$

$$= \frac{1}{2K} \int_\Omega (z^*)^2 \, dx \quad ...(55)$$

Furthermore, define the functional $J^* : B^* \times D^* \times E^* \to \mathbb{R}$ by

$$J^*(v_1^*, v_0^*, z^*) = -F_1^*(v_1^*) - F_2^*(v_1^*, v_0^*, z^*) + F_3^*(z^*)$$

and the exactly penalized functional $J_1^* : D^* \times B^* \times E^* \to \mathbb{R}$ by

$$J_1^*(v_1^*, v_0^*, z^*) = J^*(v_1^*, v_0^*, z^*)$$

$$+ \frac{1}{12\alpha K_3^2} \left\| -\gamma \nabla^2 \left(\frac{z^*}{K} \right) + 2v_0^* \left(\frac{z^*}{K} \right) - f \right\|_{0,2}^2 \quad ...(56)$$

Let $(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*) \in D^* \times B^* \times E^*$ be such that

$$\delta J^*(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*) = 0$$

From this and from the Legendre transform proprieties, for

$$u_0 = \frac{\hat{z}^*}{K} \in V_1$$

we obtain

$$\delta J(u_0) = 0,$$

$$J(u_0) = J^*(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*) = J_1^*(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*)$$

and

$$\delta J_1^*(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*) = 0$$

Observe that

$$\frac{\partial^2 J_1^*(v_1^*, v_0^*, z^*)}{\partial (z^*)^2}$$

$$= -\frac{1}{2v_0^* + K} + \frac{1}{K}$$

$$+\frac{1}{6\alpha K_3^2} \frac{(-\gamma \nabla^2 + 2v_0^*)}{K^2} \quad ...(57)$$

in $D^* \times B^* \times E^*$,

Assume $\hat{v}_0^* \in B^*$ is such that

$$\frac{\partial^2 J_1^*(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*)}{\partial (z^*)^2} > 0$$

Therefore, since J_1^* is quadratic in z^*, we obtain

$$J_1^*(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*) = \inf_{z^* \in E^*} J_1^*(\hat{v}_1^*, \hat{v}_0^*, z^*)$$

Moreover, for $(v_1^*, v_0^*) \in D^* \times B^*$, we have

$$\frac{\partial^2 J_1^*(v_1^*, v_0^*, \hat{z}^*)}{\partial (v_1^*)^2}$$

$$= -\frac{1}{-\gamma \nabla^2} - \frac{1}{2v_0^* + K}$$

$$< 0 \quad ...(58)$$

and denoting

$$u = \frac{-v_1^* + \hat{z}^*}{2v_0^* + K}$$

we have

$$\frac{\partial^2 J_1^*(v_1^*, v_0^*, \hat{z}^*)}{\partial (v_0^*)^2}$$

$$= -\frac{1}{\alpha} - \frac{4u^2}{2v_0^* + K} + \frac{2}{3\alpha K_3^2}\left(\frac{z^*}{K}\right)^2$$

$$\leq -\frac{1}{\alpha} - \frac{4u^2}{2v_0^* + K} + \frac{2}{3\alpha}$$

$$= -\frac{1}{3\alpha} - \frac{4u^2}{2v_0^* + K}$$

$$< 0 \quad \text{...(59)}$$

and

$$\frac{\partial^2 J_1^*\left(v_1^*, v_0^*, \hat{z}^*\right)}{\partial v_1^* \partial v_0^*}$$

$$= \frac{2u}{2v_0^* + K} \quad \text{...(60)}$$

From such assumptions and results, by direct computation, we may obtain

$$\det\left\{\frac{\partial^2 J_1^*\left(v_1^*, v_0^*, \hat{z}^*\right)}{\partial v_0^* \partial v_1^*}\right\} = \left(\frac{\partial^2 J_1^*\left(v_1^*, v_0^*, \hat{z}^*\right)}{\partial\left(v_1^*\right)^2}\right)\left(\frac{\partial^2 J_1^*\left(v_1^*, v_0^*, \hat{z}^*\right)}{\partial\left(v_0^*\right)^2}\right) - \left(\frac{\partial^2 J_1^*\left(v_1^*, v_0^*, \hat{z}^*\right)}{\partial v_1^* \partial v_0^*}\right)^2 > 0 \quad \text{...(61)}$$

in $D^* \times B^*$.

From such results, we may infer that $J_1^*\left(v_1^*, v_0^*, \hat{z}^*\right)$ is concave in $\left(v_1^*, v_0^*\right)$ on $D^* \times B^*$, so that

$$J_1^*\left(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*\right) = \sup_{\left(v_1^*, v_0^*\right) \in D^* \times B^*} J_1^*\left(v_1^*, v_0^*, \hat{z}^*\right)$$

From these previous results and a standard Saddle Point Theorem, we have got

$$J_1^*\left(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*\right) = \sup_{\left(v_1^*, v_0^*\right) \in D^* \times B^*} \left\{\inf_{z^* \in E^*} J_1^*\left(v_1^*, v_0^*, z^*\right)\right\}$$

$$= \inf_{z^* \in E^*} \left\{\sup_{\left(v_1^*, v_0^*\right) \in D^* \times B^*} J_1^*\left(v_1^*, v_0^*, z^*\right)\right\} \quad \text{...(62)}$$

Finally, observe that

$$J\left(u_0\right) = J_1^*\left(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*\right)$$

$$\leq F_1(u) + F_2\left(u, \hat{v}_0^*\right) - \left\langle u, z^*\right\rangle_{L^2} + F_3^*\left(z^*\right)$$

$$+ \frac{1}{12\alpha K_3^2}\left\|\left(-\gamma \nabla^2\left(\frac{z^*}{K}\right) + 2\hat{v}_0^*\left(\frac{z^*}{K}\right) - f\right)\right\|_{0,2}^2 \quad \text{...(63)}$$

$\forall u \in V_1, z^* \in E^*$

In particular, for $z^* = Ku$, we obtain

$$J(u_0) = J_1^*\left(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*\right)$$

$$\leq F_1(u) + F_2\left(u, \hat{v}_0^*\right) - F_3(u)$$

$$+\frac{1}{12\alpha K_3^2}\left\|\left(-\gamma\nabla^2 u + 2\hat{v}_0^* u - f\right)\right\|_{0,2}^2$$

$$\leq \sup_{v_0^* \in Y^*}\left\{F_1(u) + F_2\left(u, v_0^*\right) - F_3(u)\right\}$$

$$+\frac{1}{12\alpha K_3^2}\left\|\left(-\gamma\nabla^2 u + 2\hat{v}_0^* u - f\right)\right\|_{0,2}^2$$

$$= J(u) + \frac{1}{12\alpha K_3^2}\left\|\left(-\gamma\nabla^2 u + 2\hat{v}_0^* u - f\right)\right\|_{0,2}^2 \quad \text{...(64)}$$

$\forall u \in V_1$

Joining the pieces, we have got

$$J(u_0) = \inf_{u \in V_1}\left\{J(u) + \frac{1}{12\alpha K_3^2}\left\|\left(-\gamma\nabla^2 u + 2\hat{v}_0^* u - f\right)\right\|_{0,2}^2\right.$$

$$= J_1^*\left(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*\right)$$

$$= \sup_{\left(v_1^*, v_0^*\right) \in D^* \times B^*}\left\{\inf_{z^* \in E^*} J_1^*\left(v_1^*, v_0^*, z^*\right)\right\}$$

$$= \inf_{z^* \in E^*}\left\{\sup_{\left(v_1^*, v_0^*\right) \in D^* \times B^*} J_1^*\left(v_1^*, v_0^*, z^*\right)\right\} \quad \text{...(65)}$$

The objective of this section is complete.

7. A Sixth Duality Principle

Define $K_3 = 3,\ K \gg \max\{K_3, \alpha, \gamma, \beta, 1/\alpha, 1/\gamma, 1/\beta\}$ and $0 < \varepsilon \ll 1$.

Define the functionals $F_1 : V \to \mathbb{R}$, $F_2 : V \times Y^* \to \mathbb{R}$ and $F_3 : V \to \mathbb{R}$ by

$$F_1(u) = \frac{\varepsilon}{2}\int_\Omega u^2\,dx - \langle u, f\rangle_{L^2} \quad \text{...(66)}$$

$$F_2\left(u, v_0^*\right) = \frac{\gamma}{2}\int_\Omega \nabla u \cdot \nabla u\ dx + \left\langle u^2, v_0^*\right\rangle_{L^2} + \frac{K}{2}\int_\Omega u^2\,dx$$

$$-\frac{\varepsilon}{2}\int_\Omega u^2 dx - \frac{1}{2\alpha}\int_\Omega (v_0^*)^2\, dx - \beta\int_\Omega v_0^*\, dx \quad \text{...(67)}$$

$$F_3(u) = \frac{K}{2}\int_\Omega u^2 dx$$

Define also,

$$V_1 = \{u \in V : \|u\|_\infty \leq K_3 \text{ and } uf \geq 0, \text{ in } \Omega\}$$

$$B^* = \{v_0^* \in Y^* : \|2v_0^*\|_\infty \leq \sqrt[8]{K}\}$$

$$D^* = \{v_1^* \in Y^* : \|v_1^*\|_\infty \leq \sqrt[8]{K}\}$$

and

$$E^* = \{z^* \in [Y^*] : \|z^*\|_\infty \leq KK_3 \text{ and } z^* f \geq 0, \text{ in } \Omega\}$$

Moreover, define the polar functionals $F_1^* : [Y^*] \to \mathbb{R}$, $F_2^* : [Y^*]^3 \to \mathbb{R}$ and $F_3^* : Y^* \to \mathbb{R}$ by

$$F_1^*(v_1^*) = \sup_{u \in V}\{\langle u, v_1^*\rangle_{L^2} - F_1(u)\}$$

$$= \frac{1}{2}\int_\Omega \frac{(v_1^* + f)^2}{\varepsilon}\, dx \quad \text{...(68)}$$

$$F_2^*(v_1^*, v_0^*, z^*) = \sup_{u \in V}\{\langle u, -v_1^* + z^*\rangle_{L^2} - F_2(u, v_0^*)\}$$

$$= \frac{1}{2}\int_\Omega \frac{(-v_1^* + z^*)^2}{-\gamma\nabla^2 + 2v_0^* + K - \varepsilon}\, dx$$

$$+\frac{1}{2\alpha}\int_\Omega (v_0^*)^2\, dx + \beta\int_\Omega v_0^*\, dx \quad \text{...(69)}$$

if $v_0^* \in B^*$,

$$F_3^*(z^*) = \sup_{w \in L^2}\{\langle w, z^*\rangle_{L^2} - F_3(w)\}$$

$$= \frac{1}{2K}\int_\Omega (z^*)^2\, dx \quad \text{...(70)}$$

Furthermore, define the functional $J^* : B^* \times D^* \times E^* \to \mathbb{R}$ by

$$J^*(v_1^*, v_0^*, z^*) = -F_1^*(v_1^*) - F_2^*(v_1^*, v_0^*, z^*) + F_3^*(z^*)$$

and the exactly penalized functional $J_1^* : D^* \times B^* \times E^* \to \mathbb{R}$ by

$$J_1^*\left(v_1^*, v_0^*, z^*\right) = J^*\left(v_1^*, v_0^*, z^*\right)$$

$$+\frac{1}{12\alpha K_3^3}\left\|-\gamma \nabla^2\left(\frac{z^*}{K}\right) + 2v_0^*\left(\frac{z^*}{K}\right) - f\right\|_{0,2}^2 \quad ...(71)$$

Let $\left(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*\right) \in D^* \times B^* \times E^*$ be such that

$$\delta J^*\left(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*\right) = 0$$

From this and from the Legendre transform proprieties, for

$$u_0 = \frac{\hat{z}^*}{K} \in V_1$$

we obtain

$$\delta J(u_0) = 0,$$

$$J\left(u_0\right) = J^*\left(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*\right) = J_1^*\left(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*\right)$$

and

$$\delta J_1^*\left(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*\right) = 0$$

Observe that

$$\frac{\partial^2 J_1^*\left(v_1^*, v_0^*, z^*\right)}{\partial\left(z^*\right)^2}$$

$$= -\frac{1}{2v_0^* + K - \varepsilon} + \frac{1}{K}$$

$$+\frac{1}{6\alpha K_3^2}\frac{\left(-\gamma \nabla^2 + 2v_0^*\right)^2}{K^2} \quad ...(72)$$

in $D^* \times B^* \times E^*$,

Assume $\hat{v}_0^* \in B^*$ is such that

$$\frac{\partial^2 J_1^*\left(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*\right)}{\partial\left(z^*\right)^2} > 0$$

Therefore, since J_1^* is quadratic in z^*, we obtain

$$J_1^*\left(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*\right) = \inf_{z^* \in E^*} J_1^*\left(\hat{v}_1^*, \hat{v}_0^*, z^*\right)$$

Moreover, for $\left(v_1^*, v_0^*\right) \in D^* \times B^*$, we have

$$\frac{\partial^2 J_1^*\left(v_1^*, v_0^*, \hat{z}^*\right)}{\partial\left(v_1^*\right)^2}$$

$$= -\frac{1}{\varepsilon} - \frac{1}{-\gamma\nabla^2 + 2v_0^* + K - \varepsilon}$$

$$< 0 \qquad ...(73)$$

and denoting

$$u = \frac{-v_1^* + \hat{z}^*}{-\gamma\nabla^2 + 2v_0^* + K - \varepsilon}$$

we have

$$\frac{\partial^2 J_1^*\left(v_1^*, v_0^*, \hat{z}^*\right)}{\partial\left(v_0^*\right)^2}$$

$$= -\frac{1}{\alpha} - \frac{4u^2}{-\gamma\nabla^2 + 2v_0^* + K - \varepsilon} + \frac{2}{3\alpha K_3^2}\left(\frac{z^*}{K}\right)^2$$

$$\leq -\frac{1}{\alpha} - \frac{4u^2}{-\gamma\nabla^2 + 2v_0^* + K - \varepsilon} + \frac{2}{3\alpha}$$

$$= -\frac{1}{3\alpha} - \frac{4u^2}{-\gamma\nabla^2 + 2v_0^* + K - \varepsilon}$$

$$< 0 \qquad ...(74)$$

and

$$\frac{\partial^2 J_1^*\left(v_1^*, v_0^*, \hat{z}^*\right)}{\partial v_1^* \partial v_0^*}$$

$$= \frac{2u}{-\gamma\nabla^2 + 2v_0^* + K - \varepsilon} \qquad ...(75)$$

From such assumptions and results, by direct computation, we may obtain

$$\det\left\{\frac{\partial^2 J_1^*\left(v_1^*, v_0^*, \hat{z}^*\right)}{\partial v_0^* \partial v_1^*}\right\} = \left(\frac{\partial^2 J_1^*\left(v_1^*, v_0^*, \hat{z}^*\right)}{\partial\left(v_1^*\right)^2}\right)\left(\frac{\partial^2 J_1^*\left(v_1^*, v_0^*, \hat{z}^*\right)}{\partial\left(v_0^*\right)^2}\right) - \left(\frac{\partial^2 J_1^*\left(v_1^*, v_0^*, \hat{z}^*\right)}{\partial v_1^* \partial v_0^*}\right)^2 > 0 \qquad ...(76)$$

in $D^* \times B^*$.

From such results, we may infer that $J_1^*\left(v_1^*, v_0^*, \hat{z}^*\right)$ is concave in $\left(v_1^*, v_0^*\right)$ on $D^* \times B^*$, so that

$$J_1^*\left(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*\right) = \sup_{\left(v_1^*, v_0^*\right) \in D^* \times B^*} J_1^*\left(v_1^*, v_0^*, \hat{z}^*\right)$$

From these previous results and a standard Saddle Point Theorem, we have got

$$J_1^*\left(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*\right) = \sup_{\left(v_1^*, v_0^*\right) \in D^* \times B^*} \left\{ \inf_{z^* \in E^*} J_1^*\left(v_1^*, v_0^*, z^*\right) \right\}$$

$$= \inf_{z^* \in E^*} \left\{ \sup_{\left(v_1^*, v_0^*\right) \in D^* \times B^*} J_1^*\left(v_1^*, v_0^*, z^*\right) \right\} \quad ...(77)$$

Finally, observe that

$$J\left(u_0\right) = J_1^*\left(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*\right)$$

$$\leq F_1\left(u\right) + F_2\left(u, \hat{v}_0^*\right) - \left\langle u, z^* \right\rangle_{L^2} + F_3^*\left(z^*\right)$$

$$+ \frac{1}{12\alpha K_3^2} \left\| \left(-\gamma \nabla^2 \left(\frac{z^*}{K} \right) + 2\hat{v}_0^* \left(\frac{z^*}{K} \right) - f \right) \right\|_{0,2}^2 \quad ...(78)$$

$\forall u \in V_1, z^* \in E^*$

In particular, for $z^* = Ku$, we obtain

$$J\left(u_0\right) = J_1^*\left(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*\right)$$

$$\leq F_1\left(u\right) + F_2\left(u, \hat{v}_0^*\right) - F_3\left(u\right)$$

$$+ \frac{1}{12\alpha K_3^2} \left\| \left(-\gamma \nabla^2 u + 2\hat{v}_0^* u - f \right) \right\|_{0,2}^2$$

$$\leq \sup_{v_0^* \in Y^*} \left\{ F_1\left(u\right) + F_2\left(u, v_0^*\right) - F_3\left(u\right) \right\}$$

$$+ \frac{1}{12\alpha K_3^2} \left\| \left(-\gamma \nabla^2 u + 2\hat{v}_0^* u - f \right) \right\|_{0,2}^2$$

$$= J\left(u\right) + \frac{1}{12\alpha K_3^2} \left\| \left(-\gamma \nabla^2 u + 2\hat{v}_0^* u - f \right) \right\|_{0,2}^2 \quad ...(79)$$

$\forall u \in V_1$

Joining the pieces, we have got

$$J(u_0) = \inf_{u \in V_1} \left\{ J(u) + \frac{1}{12\alpha K_3^2} \left\| \left(-\gamma \nabla^2 u + 2\hat{v}_0^* u - f \right) \right\|_{0,2}^2 \right.$$

$$= J_1^*\left(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*\right)$$

$$= \sup_{\left(v_1^*, v_0^*\right) \in D^* \times B^*} \left\{ \inf_{z^* \in E^*} J_1^*\left(v_1^*, v_0^*, z^*\right) \right\}$$

$$= \inf_{z^* \in E^*} \left\{ \sup_{\left(v_1^*, v_0^*\right) \in D^* \times B^*} J_1^*\left(v_1^*, v_0^*, z^*\right) \right\} \quad \text{...(80)}$$

The objective of this section is complete.

8. A Duality Principle for the Complex Ginzburg-Landau System

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded and connected set with a regular (Lipschitzian) boundary by $\partial\Omega$.

In this section, generically we denote,

$$\langle h_1, h_2 \rangle_{L^2} = R_e \left\{ \int_\Omega h_1, h_2^*, dx \right\}, \forall h_1, h_2 \in L^2(\Omega; \mathbb{C})$$

with similar notations for vectorial cases.

Here, h_2^* denotes the complex conjugate of h_2 and $R_e[z]$ denotes the real part of $z \in \mathbb{C}$.

Now, define a complex Ginzburg-Landau type functional $J : V \to \mathbb{R}$, by

$$J(u, \mathrm{A}) = \frac{\gamma}{2} \int_\Omega \left| \nabla u - i\rho \mathrm{A} u \right|^2 du$$

$$+ \frac{\alpha}{2} \int_\Omega \left(u^2 - \beta \right)^2 dx - \langle u, f \rangle_{L^2}$$

$$+ \frac{1}{8\pi} \int_\Omega \left| \mathrm{Curl\,A\text{-}B}_0 \right|^2 dx \quad \text{...(81)}$$

Here,

$$V_1 = W_0^{1,2}(\Omega; \mathbb{C})$$

$$V_2 = W^{1,2}(\Omega; \mathbb{R}^3)$$

and

$$V = V_1 \times V_2$$

Moreover, $\gamma > 0$, $\alpha > 0$, $\beta > 0$ and $f \in L^\infty(\Omega; \mathbb{C})$. We also denote

$Y = Y^* = L^2(\Omega)$,

$$Y_2 = Y_2^* = L^2(\Omega;\mathbb{C})$$

and

$$Y_1 = Y_1^* = L^2(\Omega;\mathbb{C}^3)$$

Here $u \in W_0^{1,2}(\Omega;\mathbb{C})$ denotes the local density proportion of super-conducting electrons in the superconductive sample Ω.

Also, $\mathrm{A} : W^{1,2}(\Omega;\mathbb{R}^3)$ is a magnetic potential and $\mathrm{B}_0 \in L^2(\Omega;\mathbb{R}^3)$ denotes an external magnetic field.

Define $K_3 = 3,\ K \gg \max\{K_3, \alpha, \gamma, \beta, 1/\alpha, 1/\gamma.1/\beta\}$.

Define the functionals $F_1 : V \to \mathbb{R}$, $F_2 : V_1 \times Y^* \to \mathbb{R}$ and $F_3 : V_1 \to \mathbb{R}$ and $F_4 : V \times Y_1^* \to \mathbb{R}$ by

$$F_1(u) = \frac{\gamma}{2}\int_\Omega |\nabla u - i\rho \mathrm{A} u|^2\, dx \quad ...(82)$$

$$F_2(u, v_0^*) = \left\langle |u|^2, v_0^* \right\rangle_{L^2} + \frac{K}{2}\int_\Omega |u|^2\, dx$$

$$-\frac{1}{2\alpha}\int_\Omega (v_0^*)^2\, dx - \beta\int_\Omega v_0^*\, dx \quad ...(83)$$

$$F_3(u) = K\int_\Omega |u|^2\, dx$$

and

$$F_4(u, \mathrm{A}, v_1^*) = \left\langle \nabla u - i\rho \mathrm{A} u, v_1^* \right\rangle_{L^2}$$

$$+\frac{K}{2}\int_\Omega |u|^2\, dx - \langle u, f\rangle_{L^2} \quad ...(84)$$

Define also,

$$V_1 = \{u \in V : \|u\|_\infty \le K_3, \text{ in } \Omega\}$$

$$B^* = \left\{v_0^* \in Y^* : \|2v_0^*\|_\infty \le \sqrt[8]{K}\right\}$$

$$D_1^* = \left\{v_1^* \in Y_1^* : \|v_1^*\|_\infty \le \sqrt[8]{K}\right\}$$

$$D_2^* = \left\{v_2^* \in Y_2^* : \|v_2^*\|_\infty (3/2) K K_3\right\}$$

$$E^* = \left\{z^* \in Y_2^* : \|z^*\|_\infty \le 2KK_3, \text{in } \Omega\right\}$$

and assuming the Gauge of London

$$E_1 = \{\mathrm{A} \in V_2 : \mathrm{div}\mathrm{A} = 0, \text{ in } \Omega \text{ and } \mathrm{A} \cdot \mathrm{n} = 0, \text{ on } \partial\Omega\}$$

Moreover, define the polar functionals $F_1^* : Y_1^* \to \mathbb{R}$, $F_2^* : Y_2^* \times Y^* \times Y_2^* \to \mathbb{R}$ and $F_3^* : Y_2^* \to \mathbb{R}$ and $F_4^* : Y_2^* \times Y_1^* \times Y_2^* \times V_2 \to \mathbb{R}$ by

$$F_1^*(v_1^*) = \sup_{u \in V} \left\{ \langle w, v_1^* \rangle_{L^2} - \frac{\gamma}{2} \int_\Omega |w|^2 \, dx \right\}$$

$$= \frac{1}{2} \int_\Omega |v_1^*|^2 \, dx \quad \text{...(85)}$$

$$F_2^*(v_1^*, v_0^*, z^*) = \sup_{u \in V} \left\{ \langle u, -v_2^* + z^*/2 \rangle_{L^2} - F_2(u, v_0^*) \right\}$$

$$= \frac{1}{2} \int_\Omega \frac{|-v_2^* + z^*/2|^2}{2v_0^* + K} dx$$

$$+ \frac{1}{2\alpha} \int_\Omega (v_0^*)^2 \, dx + \beta \int_\Omega v_0^* \, dx \quad \text{...(86)}$$

if $v_0^* \in B^*$

$$F_3^*(z^*) = \sup_{w \in L^2} \left\{ \langle w, z^* \rangle_{L^2} - F_3(w) \right\}$$

$$= \frac{1}{4K} \int_\Omega (z^*)^2 \, dx \quad \text{...(87)}$$

and

$$F_4^*(v_2^*, v_1^*, z^*, \mathrm{A}) = \sup_{u \in V} \left\{ \langle u, v_2^* + z^*/2 \rangle_{L^2} - F_2(u, \mathrm{A}, v_1^*) \right\}$$

$$= \frac{1}{2} \int_\Omega \frac{|v_2^* + z^*/2 + \mathrm{div}\, v_1^* + i\rho \mathrm{A} \cdot v_1^* + f|^2}{K} dx$$

$$+ \frac{1}{2\alpha} \int_\Omega (v_0^*)^2 dx + \beta \int_\Omega v_0^* \, dx \quad \text{...(88)}$$

Furthermore, define

$$F_5(\mathrm{A}) = \frac{1}{8\pi} \int_\Omega |\mathrm{curl}\, \mathrm{A} \text{-} \mathrm{B}_0|^2 \, dx$$

define also the functional $J^* : D_1^* \times D_2^* \times B^* \times E^* \times E_1 \to \mathbb{R}$ by

$$J^*(v_2^*, v_1^*, v_0^*, z^*, \mathrm{A}) = -F_1^*(v_1^*) - F_2^*(v_2^*, v_0^*, z^*) + F_3^*(z^*) - F_4^*(v_2^*, v_1^*, z^*, \mathrm{A}) + F_5(\mathrm{A})$$

and the exactly penalized functional $J^*: D_1^* \times D_2^* \times B^* \times E^* \times E_1 \to \mathbb{R}$ by

$$J_1^*\left(v_2^*, v_1^*, v_0^*, z^*, \text{A}\right) = J^*\left(v_2^*, v_1^*, v_0^*, z^*, \text{A}\right)$$

$$+\frac{K_1}{2}\left\|-\text{div}\, v_1^* - i\rho \text{A} v_1^* + 2v_0^*\left(\frac{z^*}{2K}\right) - f\right\|_{0,2}^2 \quad \text{...(89)}$$

Let $\left(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*, \hat{\text{A}}\right) \in D_1^* \times D_2^* \times B^* \times E^* \times E_1$ be such that

$$\delta J^*\left(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*, \hat{\text{A}}\right) = 0$$

From this and from the Legendre transform proprieties, for

$$u_0 = \frac{\hat{z}^*}{2K} \in V_1$$

we obtain

$$\delta J\left(u_0, \hat{\text{A}}\right) = 0$$

$$J\left(u_0, \text{A}\right) = J^*\left(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*, \hat{\text{A}}\right) = J_1^*\left(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*, \hat{\text{A}}\right)$$

and

$$\delta J_1^*\left(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*, \hat{\text{A}}\right) = 0$$

Here $K_1 > 0$ is the largest positive (in fact close) real constant such that $J_1^*\left(v_2^*, v_1^*, v_0^*, \hat{z}^*, \hat{\text{A}}\right)$ is concave in $\left(v_2^*, v_1^*, v_0^*\right)$ in $D_1^* \times D_2^* \times B^*$.

Observe that J_1^* is quadratic in (z^*, A). Here we assume $K_1 > 0$ is also such that

$$\det\left\{\frac{\partial^2 J_1^*\left(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*, \hat{\text{A}}\right)}{\partial\left(z^*\right)\partial \text{A}}\right\} > 0$$

Therefore, since J_1^* is quadratic in (z^*, A), we obtain

$$J_1^*\left(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*, \hat{\text{A}}\right) = \inf_{\left(z^*, \text{A}\right) \in E^* \times E_1} J_1^*\left(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, z^*, \text{A}\right)$$

Also, from the previous mentioned concavity,

$$J_1^*\left(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*, \hat{\text{A}}\right) = \sup_{\left(v_2^*, v_1^*, v_0^*\right) \in D_2^* \times D_1^* \times B^*} J_1^*\left(v_2^*, v_1^*, v_0^*, \hat{z}^*, \hat{\text{A}}\right)$$

From these previous results and a standard Saddle Point Theorem, we have got

$$J_1^*\left(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*, \hat{\mathrm{A}}\right) = \sup_{\left(v_2^*, v_1^*, v_0^*\right) \in D_2^* \times D_1^* \times B^*} \left\{ \inf_{\left(z^*, \mathrm{A}\right) \in E^* \times E_1} J_1^*\left(v_2^*, v_1^*, v_0^*, z^*, \mathrm{A}\right) \right\}$$

$$= \inf_{\left(z^*, \mathrm{A}\right) \in E^* \times E_1} \left\{ \sup_{\left(v_2^*, v_1^*, v_0^*\right) \in D_2^* \times D_1^* \times B^*} J_1^*\left(v_2^*, v_1^*, v_0^*, z^*, \mathrm{A}\right) \right\} \quad \text{...(90)}$$

Finally, observe that

$$J\left(u_0, \hat{\mathrm{A}}\right) = J_1^*\left(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*, \hat{\mathrm{A}}\right)$$

$$\leq F_1\left(u, \mathrm{A}\right) + F_2\left(u, \hat{v}_0^*\right) - \left\langle u, z^* \right\rangle_{L^2} + F_3^*\left(z^*\right) + \frac{K}{2} \int_\Omega |u|^2 \, dx$$

$$+F_5\left(\mathrm{A}\right) + \frac{K_1}{2} \left\| -\operatorname{div} \hat{v}_1^* - i\rho \mathrm{A} \hat{v}_1^* + 2\hat{v}_0^* \left(\frac{z^*}{2K} \right) - f \right\|_{0,2}^2 \quad \text{...(91)}$$

$\forall u \in V_1, z^* \in E^*, \mathrm{A} \in E_1$.

In particular, for $z^* = 2Ku$, we obtain

$$J\left(u_0, \hat{\mathrm{A}}\right) = J_1^*\left(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*, \hat{\mathrm{A}}\right)$$

$$\leq F_1\left(u, \mathrm{A}\right) + F_2\left(u, \hat{v}_0^*\right) - F_3\left(u\right) + \frac{K}{2} \int_\Omega |u|^2 \, dx$$

$$+F_5\left(\mathrm{A}\right) + \frac{K_1}{2} \left\| -\operatorname{div} \hat{v}_1^* - i\rho \mathrm{A} \hat{v}_1^* + 2\hat{v}_0^* u - f \right\|_{0,2}^2$$

$$\leq \sup_{v_0^* \in Y^*} \left\{ F_1\left(u\right) + F_2\left(u, v_0^*\right) - F_3\left(u\right) + \frac{K}{2} \int_\Omega |u|^2 \, dx \right\}$$

$$+F_5\left(\mathrm{A}\right) + \frac{K_1}{2} \left\| -\operatorname{div} \hat{v}_1^* - i\rho \mathrm{A} \hat{v}_1^* + 2\hat{v}_0^* u - f \right\|_{0,2}^2$$

$$= J\left(u, \mathrm{A}\right) + \frac{K_1}{2} \left\| -\operatorname{div} \hat{v}_1^* - i\rho \mathrm{A} \hat{v}_1^* + 2\hat{v}_0^* u - f \right\|_{0,2}^2 \quad \text{...(92)}$$

$\forall u \in V_1$

Joining the pieces, we have got

$$J\left(u_0, \hat{\mathrm{A}}\right) = \inf_{(u, \mathrm{A}) \in V_1 \times E_1} \left\{ J\left(u, \mathrm{A}\right) + \frac{K_1}{2} \left\| -\operatorname{div} \hat{v}_1^* - i\rho \mathrm{A} \hat{v}_1^* + 2\hat{v}_0^* u - f \right\|_{0,2}^2 \right\}$$

$$= J_1^*\left(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{z}^*, \hat{\mathrm{A}}\right)$$

$$= \sup_{(v_2^*, v_1^*, v_0^*) \in D_2^* \times D_1^* \times B^*} \left\{ \inf_{(z^*, \mathrm{A}) \in E^* \times E_1} J_1^*\left(v_2^*, v_1^*, v_0^*, z^*, \mathrm{A}\right) \right\}$$

$$= \inf_{(z^*, \mathrm{A}) \in E^* \times E_1} \left\{ \sup_{(v_2^*, v_1^*, v_0^*) \in D_2^* \times D_1^* \times B^*} J_1^*\left(v_2^*, v_1^*, v_0^*, z^*, \mathrm{A}\right) \right\} \quad \text{...(93)}$$

The objective of this section is complete.

9. An Eighth Duality Principle

Define $K_3 = 3$ and $K > 0, K_1 > 0, K_2 > 0$ such that

$$K_2 \gg K_1 \gg K \gg \max\{K_3, \alpha, \gamma, \beta, 1/\alpha, 1/\gamma, 1/\beta\}.$$

Define the functionals $F_1 : V \to \mathbb{R}$, $F_2 : V \to \mathbb{R}$, $F_3 : V \times Y^* \to \mathbb{R}$ and $F_4 : V \times Y^* \to \mathbb{R}$, by

$$F_1(u) = \frac{\gamma}{2}\int_\Omega \nabla u \cdot \nabla u \; dx + \frac{K_1}{2}\int_\Omega u^2 dx - \langle u, f \rangle_{L^2} \quad \text{...(94)}$$

$$F_2(u) = -\frac{K_1}{2}\int_\Omega u^2 dx \quad \text{...(95)}$$

$$F_3(u, v_0^*) = \frac{K}{2}\int_\Omega u^2 dx - \frac{1}{2\alpha}\int_\Omega (v_0^*)^2 \, dx - \beta\int_\Omega v_0^* dx$$

and

$$F_4(u, v_0^*) = -\langle u^2, v_0^* \rangle_{L^2} + \frac{K}{2}\int_\Omega u^2 dx$$

Define also,

$$V_1 = \{u \in V : \|u\|_\infty \le K_3 \text{ and } uf \ge 0, \text{ in } \Omega\}$$

$$B^* = \left\{v_0^* \in Y^* : \|2v_0^*\|_\infty \le \sqrt[8]{K}\right\}$$

$$D^* = \left\{(v_1^*, v_2^*) \in Y^* \times Y^* : \|v_1^*\|_\infty \le (3/2)K_1K_3 \text{ and } \|v_2^*\|_\infty \le (3/2)K_1K_3\right\}$$

and

$$E^* = \left\{z^* \in Y^* : \|z^*\|_\infty \le (3/2)KK_3 \text{ and } z^* f \ge 0, \text{ in } \Omega\right\}$$

Moreover, define the polar functionals $F_1^* : [Y^*]^2 \to \mathbb{R}$, $F_2^* : [Y^*] \to \mathbb{R}$, $F_3^* : [Y^*]^3 \to \mathbb{R}$ and $F_4^* : [Y^*]^2 \to \mathbb{R}$ by

$$F_1^*(v_1^*, z^*) = \sup_{u \in V}\left\{\langle u, v_1^* + z^* \rangle_{L^2} - F_1(u)\right\}$$

$$= \frac{1}{2}\int_\Omega \frac{\left(v_1^* + z^* + f\right)^2}{-\gamma \nabla^2 + K_1} dx \quad ...(96)$$

$$F_2^*\left(v_2^*\right) = \inf_{u \in V}\left\{\left\langle u, v_2^*\right\rangle_{L^2} - F_2\left(u\right)\right\}$$

$$= \frac{1}{2}\int_\Omega \frac{\left(v_2^*\right)^2}{-K_1} dx \quad ...(97)$$

and

$$F_3^*\left(v_1^*, v_2^*, v_0^*\right) = \sup_{u \in V}\left\{\left\langle u, -v_1^* - v_2^*\right\rangle_{L^2} - F_3\left(u, v_0^*\right)\right\}$$

$$= \frac{1}{2K}\int_\Omega \left(v_1^* + v_2^*\right)^2 dx$$

$$+ \frac{1}{2\alpha}\int_\Omega \left(v_0^*\right)^2 dx + \beta \int_\Omega v_0^* dx \quad ...(98)$$

and

$$F_4^*\left(z^*\right) = \sup_{u \in V}\left\{\left\langle u, \mathrm{z}^*\right\rangle_{L^2} - F_3\left(u, v_0^*\right)\right\}$$

$$= \frac{1}{2}\int_\Omega \frac{\left(z^*\right)^2}{-2v_0^* + K} dx \quad ...(99)$$

Furthermore, define the functional $J^* : B^* \times D^* \times E^* \to \mathbb{R}$ by

$$J^*\left(v_1^*, v_2^*, v_0^*, z^*\right) = -F_1^*\left(v_1^*, z^*\right) - F_2^*\left(v_2^*\right) + F_3^*\left(v_1^*, v_2^*, v_0^*\right) + F_4^*\left(z^*, v_0^*\right)$$

and the exactly penalized functional $J_1^* : D^* \times B^* \times E^* \to \mathbb{R}$ by

$$J_1^*\left(v_1^*, v_2^*, v_0^*, z^*\right) = J^*\left(v_1^*, v_2^*, v_0^*, z^*\right)$$

$$-\frac{K_1^2}{6K}\left\| -\frac{v_1^* + z^* + f}{-\gamma \nabla^2 + K_1} + \frac{v_1^* + v_2^*}{K} \right\|_{0,2}^2$$

$$-\frac{K_2 K_1^2}{2}\left\| -\frac{v_1^* + v_2^*}{K} + \frac{v_2^*}{-K_1} \right\|_{0,2}^2 \quad ...(100)$$

Let $\left(\hat{v}_1^*, \hat{v}_2^*, \hat{v}_0^*, \hat{z}^*\right) \in D^* \times B^* \times E^*$ be such that

$$\delta J^*\left(\hat{v}_1^*, \hat{v}_2^*, \hat{v}_0^*, \hat{z}^*\right) = 0$$

From this and from the Legendre transform proprieties, for

$$u_0 = \frac{\hat{v}_2^*}{-K_1} \in V_1$$

we obtain

$$\delta J(u_0) = 0,$$

$$J(u_0) = J^*\left(\hat{v}_1^*, \hat{v}_2^*, \hat{v}_0^*, \hat{z}^*\right) = J_1^*\left(\hat{v}_1^*, \hat{v}_2^*, \hat{v}_0^*, \hat{z}^*\right)$$

and

$$\delta J_1^*\left(\hat{v}_1^*, \hat{v}_2^*, \hat{v}_0^*, \hat{z}^*\right) = 0$$

Observe that for $K_2 \gg K_1 \gg K > 0$ as previously specified, we have that J_1^* is concave in $\left(v_1^*, v_2^*, v_0^*\right)$ and convex in z^*, in $D^* \times B^* \times E^*$.

Therefore, since J_1^* is quadratic in z^*, we obtain

$$J_1^*\left(\hat{v}_1^*, \hat{v}_2^*, \hat{v}_0^*, \hat{z}^*\right) = \inf_{z^* \in E^*} J_1^*\left(\hat{v}_1^*, \hat{v}_2^*, \hat{v}_0^*, z^*\right)$$

Moreover, from the mentioned concavity in $\left(v_1^*, v_2^*, v_0^*\right)$ we have

$$J_1^*\left(\hat{v}_1^*, \hat{v}_2^*, \hat{v}_0^*, \hat{z}^*\right) = \sup_{\left(v_1^*, v_2^*, v_0^*\right) \in D^* \times B^*} J_1^*\left(v_1^*, v_2^*, v_0^*, \hat{z}^*\right)$$

From these previous results and a standard Saddle Point Theorem, we have got

$$J_1^*\left(\hat{v}_1^*, \hat{v}_2^*, \hat{v}_0^*, \hat{z}^*\right) = \sup_{\left(v_1^*, v_2^*, v_0^*\right) \in D^* \times B^*} \left\{ \inf_{z^* \in E^*} J_1^*\left(v_1^*, v_2^*, v_0^*, z^*\right) \right\}$$

$$= \inf_{z^* \in E^*} \left\{ \sup_{\left(v_1^*, v_2^*, v_0^*\right) \in D^* \times B^*} J_1^*\left(v_1^*, v_2^*, v_0^*, z^*\right) \right\} \quad \text{...(101)}$$

Finally, observe that

$$J(u_0) = J_1^*\left(\hat{v}_1^*, \hat{v}_2^*, \hat{v}_0^*, \hat{z}^*\right)$$

$$\leq F_1(u) + F_2^*\left(\hat{v}_2^*\right) - \left\langle u, \hat{v}_2^* \right\rangle_{L^2} - \left\langle u, z^* \right\rangle_{L^2} + F_3\left(u, \hat{v}_0^*\right) + F_4^*\left(z^*, \hat{v}_0^*\right) \quad \text{...(102)}$$

$$\forall u \in V_1, z^* \in E^*$$

In particular, for $z^* = -2\hat{v}_0^* u + Ku$ and recalling that

$$u_0 = \frac{\hat{v}_2^*}{-K_1},$$

we obtain

$$J(u_0) = J_1^*(\hat{v}_1^*, \hat{v}_2^*, \hat{v}_0^*, \hat{z}^*)$$

$$\leq F_1(u) - F_2^*(\hat{v}_2^*) + \langle u, \hat{v}_2^* \rangle_{L^2} + F_3(u, \hat{v}_0^*) - F_4(u, \hat{v}_0^*)$$

$$\leq F_1(u) - F_2^*(\hat{v}_2^*) + \langle u, \hat{v}_2^* \rangle_{L^2}$$

$$+ \sup_{v_0^* \in Y^*} \left\{ F_3(u, v_0^*) - F_4(u, v_0^*) \right\}$$

$$= J(u) + \frac{K_1}{2} \|u - u_0\|_{0,2}^2 \qquad ...(103)$$

$\forall u \in V_1$

Joining the pieces, we have got

$$J(u_0) = \inf_{u \in V_1} \left\{ J(u) + \frac{K_1}{2} \|u - u_0\|_{0,2}^2 \right.$$

$$= J_1^*(\hat{v}_1^*, \hat{v}_2^*, \hat{v}_0^*, \hat{z}^*)$$

$$= \sup_{(v_1^*, v_2^*, v_0^*) \in D^* \times B^*} \left\{ \inf_{z^* \in E^*} J_1^*(v_1^*, v_2^*, v_0^*, z^*) \right\}$$

$$= \inf_{z^* \in E^*} \left\{ \sup_{(v_1^*, v_2^*, v_0^*) \in D^* \times B^*} J_1^*(v_1^*, v_2^*, v_0^*, z^*) \right\} \qquad ...(104)$$

The objective of this section is complete.

10. A Ninth Duality Principle

Define $K_3 = 3$ and $K > 0$, $K_1 > 0$ and $K_2 > 0$ be such that

$$K_2 \gg K_1 \gg K \gg \max\{K_3, \alpha, \gamma, \beta, 1/\alpha, 1/\gamma, 1/\beta\}.$$

Define the functionals $F_1 : V \to \mathbb{R}$, $F_2 : V \times Y^* \to \mathbb{R}$, $F_3 : V \to \mathbb{R}$ and $F_4 : V \to \mathbb{R}$, by

$$F_1(u) = \frac{\gamma}{2} \int_\Omega \nabla u \cdot \nabla u \, dx + \frac{K}{2} \int_\Omega u^2 dx - \langle u, f \rangle_{L^2} \qquad ...(105)$$

$$F_2(u, v_0^*) = \langle u^2, v_0^* \rangle_{L^2} + \frac{K}{2} \int_\Omega u^2 dx$$

$$- \frac{1}{2\alpha} \int_\Omega (v_0^*)^2 \, dx - \beta \int_\Omega v_0^* dx \qquad ...(106)$$

$$F_3(u) = \frac{(K_1 + K)}{2} \int_\Omega u^2 dx \qquad ...(107)$$

and

$$F_4(u) = \frac{(-K_1 + K)}{2} \int_\Omega u^2 dx$$

Define also,

$$V_1 = \left\{ u \in V : \|u\|_\infty \le K_3 \text{ and } uf \ge 0, \text{ in } \Omega \right\}$$

$$B^* = \left\{ v_0^* \in Y^* : \|2v_0^*\|_\infty \le \sqrt[8]{K} \right\}$$

$$D^* = \left\{ v_1^* \in Y^* : \|v_1^*\|_\infty \le (3/2) K_1 K_3 \right\}$$

and

$$E^* = \left\{ z^* \in Y^* : \|z^*\|_\infty \le K K_3 \text{ and } z^* f \ge 0, \text{ in } \Omega \right\}$$

$$E_1^* = \left\{ z_1^* \in Y^* : \|z_1^*\|_\infty \le K_1 K_3 \text{ and } z_1^* f \ge 0, \text{ in } \Omega \right\}$$

Moreover, define the polar functionals $F_1^* : [Y^*]^2 \to \mathbb{R}$, $F_2^* : [Y^*]^3 \to \mathbb{R}$, $F_3^* : [Y^*]^2 \to \mathbb{R}$ and $F_4^* : [Y^*]^2 \to \mathbb{R}$ by

$$F_1^*(v_1^*, z^*) = \sup_{u \in V} \left\{ \langle u, v_1^* + z^* \rangle_{L^2} - F_1(u) \right\}$$

$$= \frac{1}{2} \int_\Omega \frac{(v_1^* + z^* + f)^2}{-\gamma \nabla^2 + K} dx \quad ...(108)$$

$$F_2^*(v_1^*, v_0^*, z^*) = \inf_{u \in V} \left\{ \langle u, -v_1^* + z^* \rangle_{L^2} - F_2(u, v_0^*) \right\}$$

$$= \frac{1}{2} \int_\Omega \frac{(-v_1^* + z^*)^2}{2v_0^* + K} dx$$

$$+ \frac{1}{2\alpha} \int_\Omega (v_0^*)^2 dx + \beta \int_\Omega v_0^* dx \quad ...(109)$$

if $v_0^* \in B^*$, and

$$F_3^*(z^*, z_1^*) = \sup_{u \in V} \left\{ \langle u, z^* + z_1^* \rangle_{L^2} - F_3(u) \right\}$$

$$= \frac{1}{2(K_1 + K)} \int_\Omega (z^* + z_1^*)^2 dx \quad ...(110)$$

and

$$F_4^*\left(z^*, z_1^*\right) = \inf_{u \in V}\left\{\left\langle u, z^* - z_1^*\right\rangle_{L^2} - F_4\left(u\right)\right\}$$

$$= \frac{1}{2\left(-K_1^* + K\right)}\int_\Omega \left(z^* - z_1^*\right)^2 dx \quad ...(111)$$

Furthermore, define the functional $J^* : D^* \times B^* \times E_1^* \times E_2^* \to \mathbb{R}$ by

$$J^*\left(v_1^*, v_0^*, z^*, z_1^*\right) = -F_1^*\left(v_1^*, z^*\right) - F_2^*\left(v_1^*, v_0^*, z^*\right) + F_3^*\left(z^*, z_1^*\right) + F_4^*\left(z^*, z_1^*\right)$$

and the exactly penalized functional $J_1^* : D^* \times B^* \times E_1^* \times E_2^* \to \mathbb{R}$ by

$$J_1^*\left(v_1^*, v_0^*, z^*, z_1^*\right) = J^*\left(v_1^*, v_0^*, z^*, z_1^*\right)$$

$$+\frac{K_2}{2}\left\|\frac{z^*}{K} - \frac{z_1^*}{K_1}\right\|_{0,2}^2$$

$$+\frac{1}{12\alpha K_3^2}\left\|-\gamma \nabla^2\left(\frac{z^*}{K}\right) + 2v_0^*\left(\frac{z^*}{K}\right) - f\right\|_{0,2}^2 \quad ...(112)$$

Let $\left(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*, \hat{z}_1^*\right) \in D^* \times B^* \times E_1^* \times E_2^*$ be such that

$$\delta J^*\left(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*, \hat{z}_1^*\right) = 0$$

From this and from the Legendre transform proprieties, for

$$u_0 = \frac{\hat{z}^* + \hat{z}_1^*}{K_1 + K} = \frac{\hat{z}_1^*}{K_1} = \frac{\hat{z}^*}{K} \in V_1$$

we obtain

$$\delta J(u_0) = 0,$$

$$J\left(u_0\right) = J^*\left(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*, \hat{z}_1^*\right) = J_1^*\left(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*, \hat{z}_1^*\right)$$

and

$$\delta J_1^*\left(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*, \hat{z}_1^*\right) = 0$$

Observe that for $K_1 \gg K > 0$ as previously specified, we have

$$\frac{\partial^2 J_1^*\left(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*, \hat{z}_1^*\right)}{\partial\left(z^*\right)^2} = -\frac{1}{-\gamma \nabla^2 + K} - \frac{1}{2\hat{v}_0^* + K}$$

$$\frac{1}{K_1+K}+\frac{1}{-K_1+K}+\frac{K_2}{K^2}+\frac{1}{6\alpha K_3^2 K^2}\left(-\gamma\nabla^2+2\hat{v}_0^*\right)^2$$

$$>0 \quad ...(113)$$

$$\frac{\partial^2 J_1^*\left(\hat{v}_1^*,\hat{v}_0^*,\hat{z}^*,\hat{z}_1^*\right)}{\partial\left(z_1^*\right)^2}=\frac{1}{K_1+K}+\frac{1}{-K_1+K}+\frac{K_2}{K_1^2}$$

$$>0 \quad ...(114)$$

and

$$\frac{\partial^2 J_1^*\left(\hat{v}_1^*,\hat{v}_0^*,\hat{z}^*,\hat{z}_1^*\right)}{\partial z^*\partial\left(z_1^*\right)}=\frac{1}{K_1+K}+\frac{1}{K_1-K}+\frac{K_2}{K_1 K} \quad ...(115)$$

Through such results, assuming $\hat{v}_0^*\in B^*$ is such that

$$B_5=\frac{2\left(-\gamma\nabla^2+2\hat{v}_0^*\right)}{\left(-\gamma\nabla^2+K\right)\left(2\hat{v}_0^*+K\right)}+\frac{\left(-\gamma\nabla^2+2\hat{v}_0^*\right)^2}{\left(6\alpha K_3^2K^2\right)}\geq\frac{1}{K^2\sqrt{K}}I_d$$

we obtain

$$\det\left\{\frac{\partial^2 J_1^*\left(\hat{v}_1^*,\hat{v}_0^*,\hat{z}^*,\hat{z}_1^*\right)}{\partial z^*\partial\left(z_1^*\right)}\right\}=\left(\frac{\partial^2 J_1^*\left(\hat{v}_1^*,\hat{v}_0^*,\hat{z}^*,\hat{z}_1^*\right)}{\partial\left(z^*\right)^2}\right)\left(\frac{\partial^2 J_1^*\left(\hat{v}_1^*,\hat{v}_0^*,\hat{z}^*,\hat{z}_1^*\right)}{\partial\left(z_1^*\right)^2}\right)-\left(\frac{\partial^2 J_1^*\left(\hat{v}_1^*,\hat{v}_0^*,\hat{z}^*,\hat{z}_1^*\right)}{\partial z^*\partial\left(z_1^*\right)}\right)^2$$

$$=\mathcal{O}\left(\frac{B_5K_2}{K_1^2}\right)$$

$$>0 \quad ...(116)$$

From such a result, since J_1^* is quadratic in $\left(z^*,z_1^*\right)$, we have

$$J_1^*\left(\hat{v}_1^*,\hat{v}_0^*,\hat{z}^*,\hat{z}_1^*\right)=\inf_{\left(z^*,z_1^*\right)\in E_1\times E_2} J_1^*\left(\hat{v}_1^*,\hat{v}_0^*,z^*,\hat{z}_1^*\right)$$

On the other hand

J_1^* is concave in $\left(v_1^*,v_0^*\right)$ in $D^*\times B^*\times E_1^*\times E_2^*$ so that

$$J_1^*\left(\hat{v}_1^*,\hat{v}_0^*,\hat{z}^*,\hat{z}_1^*\right)=\sup_{\left(v_1^*,v_0^*\right)\in D^*\times B^*} J_1^*\left(v_1^*,v_0^*,\hat{z}^*,z_1^*\right)$$

From such results and a standard Saddle Point Theorem we may infer that

$$J_1^*\left(\hat{v}_1^*,\hat{v}_0^*,\hat{z}^*,\hat{z}_1^*\right)=\inf_{\left(z^*,z_1^*\right)\in E_1\times E_2}\left\{\sup_{\left(v_1^*,v_0^*\right)\in D^*\times B^*} J_1^*\left(v_1^*,v_0^*,z^*,z_1^*\right)\right\}$$

$$= \sup_{(v_1^*, v_0^*, z_1^*) \in D^* \times B^*} \left\{ \inf_{(z^*, z_1^*) \in E_1 \times E_2} J_1^*\left(v_1^*, v_0^*, z^*, z_1^*\right) \right\} \qquad ...(117)$$

Finally, observe that

$$J(u_0) = J_1^*\left(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*, \hat{z}_1^*\right)$$

$$\leq F_1(u) + F_2\left(u, \hat{v}_0^*\right) - 2\left\langle u, z^* \right\rangle_{L^2} + F_3^*\left(z^* + z_1^*\right) + \left\langle u, z^* - z_1^* \right\rangle_{L^2} - F_4(u)$$

$$+ \frac{K_2}{2} \left\| \frac{z^*}{K} - \frac{z_1^*}{K_1} \right\|_{0,2}^2$$

$$+ \frac{1}{12\alpha K_3^2} \left\| -\gamma \nabla^2 \left(\frac{z^*}{K} \right) + 2\hat{v}_0^* \left(\frac{z^*}{K} \right) - f \right\|_{0,2}^2 \qquad ...(118)$$

$$\forall u \in V_1, z^* \in E_1^*, z_1^* \in E_2$$

In particular, for $z^* \in E_1^*$ and $z_1^* \in E_2$ such that

$$z^* = Ku$$

and

$$z_1^* = K_1 u$$

we obtain

$$J(u_0) = J_1^*\left(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*, \hat{z}_1^*\right)$$

$$= \frac{\gamma}{2} \int_\Omega \nabla u \cdot \nabla u \; dx - \left\langle u, f \right\rangle_{L^2}$$

$$+ \left\langle u^2, \hat{v}_0^* \right\rangle_{L^2} - \frac{1}{2\alpha} \int_\Omega \left(\hat{v}_0^*\right)^2 dx - \beta \int_\Omega \hat{v}_0^* \, dx$$

$$+ \frac{1}{12\alpha K_3^2} \left\| -\gamma \nabla^2 u + 2\hat{v}_0^* u - f \right\|_{0,2}^2$$

$$\leq \frac{\gamma}{2} \int_\Omega \nabla u \cdot \nabla u \; dx - \left\langle u, f \right\rangle_{L^2}$$

$$+ \sup_{v_0^* \in Y^*} \left\{ \left\langle u^2, v_0^* \right\rangle_{L^2} - \frac{1}{2\alpha} \int_\Omega \left(v_0^*\right)^2 dx - \beta \int_\Omega v_0^* \, dx \right\}$$

$$+ \frac{1}{12\alpha K_3^2} \left\| -\gamma \nabla^2 u + 2\hat{v}_0^* u - f \right\|_{0,2}^2$$

$$= \frac{\gamma}{2}\int_\Omega \nabla u \cdot \nabla u \, dx + \frac{\alpha}{2}\int \left(u^2 - \beta\right)^2 dx - \langle u, f \rangle_{L^2}$$

$$+ \frac{1}{12\alpha K_3^2}\left\| -\gamma \nabla^2 u + 2\hat{v}_0^* u - f \right\|_{0,2}^2$$

$$= J(u) + \frac{1}{12\alpha K_3^2}\left\| -\gamma \nabla^2 u + 2\hat{v}_0^* u - f \right\|_{0,2}^2 \quad \text{...(119)}$$

$\forall u \in V_1$

Joining the pieces, we have got

$$J(u_0) = \inf_{u \in V_1}\left\{ J(u) + \frac{1}{12\alpha K_3^2}\left\| -\gamma \nabla^2 u + 2\hat{v}_0^* u - f \right\|_{0,2}^2 \right.$$

$$= J_1^*\left(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*, \hat{z}_1^*\right)$$

$$= \inf_{\left(z^*, z_1^*\right) \in E_1 \times E_2}\left\{ \sup_{\left(v_1^*, v_0^*\right) \in D^* \times B^*} J_1^*\left(v_1^*, v_0^*, z^*, z_1^*\right) \right\}$$

$$= \sup_{\left(v_1^*, v_0^*\right) \in D^* \times B^*}\left\{ \inf_{\left(z^*, z_1^*\right) \in E_1 \times E_2} J_1^*\left(v_1^*, v_0^*, z^*, z_1^*\right) \right\} \quad \text{...(120)}$$

The objective of this section is complete.

11. Conclusion

In this article, through a D.C. approach, we have developed duality principles and related convex dual variational formulations suitable for an originally non-convex primal ones. As a first application, we have set a duality principle and respective convex dual formulation for a Ginzburg-Landau type equation.

We highlight the results here obtained are applicable to a large class of models in the calculus of variations, including some plate and shell non-linear theories, other models in superconduc- tivity, phase transition and micro-magnetism, among many others.

In a near future research we intend to apply such results to some of these mentioned related models.

Conflicts of Interest

The author declares no conflict of interest concerning this article.

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