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Dynamics of Complexity on Evolution of A Predator-Prey System Subject To Allee Effect on Prey Population

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Abstract
The intricacy of the development of a discrete-time prey-predator system impacted by the Allee effect on the prey population has been thoroughly examined. The positive correlation between population density and overall individual fitness is known as the Allee effect, and it is a biological phenomenon. A discrete predator-prey system of the Lotka-Volterra type is the model taken into consideration in this work, assuming that interactions between two populations are non-overlapping generations. The system's fixed points were determined, and a thorough stability analysis of the fixed points was conducted. Both with and without the Allee effect, complex bifurcation patterns were seen in every instance.
In bifurcation diagrams, the emergence of several periodic windows within chaotic regions is also understood as showing multi-periodic orbits breaking into phenomena of chaos addition and period doubling, bi-stability, intermittency character, etc. These demonstrate that the system being examined is a complicated system. In order to determine regular and chaotic attractors, calculate Lyapunov exponents for regular and chaotic motions, and calculate topological entropies—which give the system's complexity—numerical simulation work was expanded. To determine the correlation dimensions of chaotic attractors, numerical computations were further expanded. Interesting images and a tabulated correlation dimension corroborate the study's findings. This study is important for bioscience and the issues surrounding harvesting phenomena in predator-prey relationships.

Key Words: Chaos, Lyapunov Exponents, Bifurcation, Topological Entropy.

2020 Mathematical Sciences Classification: 34D45, 37H15, 34C23, 65P20

1. Introduction:

Chaotic evolutions in nonlinear systems, modelled in various disciplines, discussed extensively through articles appearing journals. Because of their internal makeup, biological systems and many others show a great deal of variability in their properties over the course of evolution. Throughout evolution, these systems, which are internally made up of several components, create ecological imbalance and disorder or discord. This is due to the fact that various components of the internal multicomponent exhibit unique features since they rarely adhere to any clear rules throughout evolution. These kinds of systems are referred to as complex systems. Because of the interactions between various agents in the system, a complex system can display both chaos in certain parameter space and other mixed phenomena known as complexity during evolution. Multiple attractors coexisting, bi-stability, intermittency, cascading effects, hysteresis qualities, etc. are examples of these characteristics. Therefore, one way to think of complexity is as its systematic nonlinear

features [1–14]. Knowing the dynamics that arise from a group of interacting pieces is the term used to describe the study of complexity.

A complex system's regular and chaotic motion as well as the dynamics brought forth by complexity must be examined while examining its evolutionary dynamics. A small number of the articles in the literature on predator-prey evolution models that are mentioned here have different assumptions based on the species and their living situations [13, 15–24]. The most suitable way to quantify chaotic evolution is with Lyapunov exponents (LCEs) [7, 8, 14, 23–43], so that at any given system state, if the LCE > 0, the evolution is chaotic, and if the LCE < 0, the development is regular. A system's complexity is determined by its topological entropy increment; a higher rise in topological entropy indicates a more complicated system [7, 12, 14, 34–42]. In fact, topological entropy quantifies the exponential rate of increase in the number of distinct orbits over time. The results of these investigations are highly valuable, fascinating, and important.

A biological phenomenon is known as the Allee effect, which is the positive correlation between population density and overall individual fitness, might affect a predator-prey system. Recent papers [43–49] have examined predator-prey systems that are susceptible to the Allee effect, which can affect either prey or predator populations. These systems are intriguing.

Investigating chaos and complexity behavior in a predator-prey system is operating in discrete time while taking into account the Allee impact on the prey population is the aim of the current study. The model is based on the well-known Lotka-Volterra system and assumes that interactions between two populations are between generations that do not overlap. The stability criteria for the system's fixed points were examined, both with and without the Allee effect. Bifurcation scenarios were appropriately evaluated and explained, and complex patterns of bifurcations were generated by modifying specific parameters in each case. Appropriate numerical calculations for both scenarios yield regular and chaotic attractors for various parameter sets. Lyapunov exponents, topological entropies, and correlation dimensions of chaotic attractors were obtained through further numerical simulation.

2. Model Description: Fixed points and their stability criteria

The basic predator-prey model put out in the Lotka-Volterra system is expressed as follows, using N and P as the densities of prey and predator populations:

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - \varepsilon NP,$$
$$\frac{dP}{dT} = bNP - \delta P. \qquad (2.1)$$

Predator birth and death rates are measured by the parameters b and stand, respectively, while the intrinsic rate of growth and carrying capacity of the prey population are represented by the parameters r and k. It is possible to convert the continuous model mentioned above into a discrete model by applying the split differences notion.

$$\frac{dN}{dt} = \frac{N_{t+h} - N_t}{h},$$
$$\frac{dP}{dt} = \frac{P_{t+h} - P_t}{h}, \qquad (2.2)$$

The matching discrete model was found to be

$$N_{t+1} = N_t + rN_t\left(1 - \frac{N_t}{k}\right) - \varepsilon N_t P_t$$
$$P_{t+1} = bN_t P_t + (1 - d)P_t. \qquad (2.3)$$

Assuming that predators are forced to perish after one generation (d = 1), we rescale the generation time h to 1 and substitute n for t. Then, when equation (2.3) has been sealed, and by replacing

$$x_t = \frac{N_t}{k},\ y_t = \frac{\varepsilon P_t}{bk},\ c = bk$$

Lastly, the discrete formulation form of equation (2.3) is changed to:

$$x_{n+1} = x_n + rx_n(1 - x_n) - cx_n y_n,$$
$$y_{n+1} = c\, x_n\, y_n \qquad (2.4)$$

Prey and predator population densities at generation n are represented by x_n and y_n in this discrete formulation, where r and c are positive factors.

In this discrete formulation, x_n and y_n represent densities of prey and predator populations at generation n, r and c are positive parameters.

The Jacobian matrix of system (2.4) obtained as

$$J = \begin{pmatrix} 1 + r - 2rx - cy & -cx \\ cy & cx \end{pmatrix} \quad (2.5)$$

System (2.4) fixed points are found as

$$P_1^* = (0,0), \quad P_2^* = (1,0) \text{ and } P_3^* = \left(\frac{1}{c}, \frac{(c-1)r}{c^2}\right).$$

Then, performing stability analysis, we find (i) the fixed point $P_1^* = (0,0)$ is unstable and (ii) the fixed point $P_2^* = (1,0)$ is stable if $c < 1$ and $r > 2$ otherwise it is unstable. For stability of fixed point $P_3^* = \left(\frac{1}{c}, \frac{(c-1)r}{c^2}\right)$, eigenvalues corresponding to this point are obtained as

$$\lambda_1 = \frac{2c^2 - cr + \sqrt{4c^3r - 4c^4r + c^2r^2}}{2c^2}, \quad \lambda_2 = \frac{2c^2 - cr - \sqrt{4c^3r - 4c^4r + c^2r^2}}{2c^2}$$

This implies $P_3^* = \left(\frac{1}{c}, \frac{(c-1)r}{c^2}\right)$ is stable only if both

$$|\lambda_1| < 1 \text{ i.e. } \left|\frac{2c^2 - cr + \sqrt{4c^3r - 4c^4r + c^2r^2}}{2c^2}\right| < 1 \text{ and}$$

$$|\lambda_2| < 1 \text{ i.e. } \left|\frac{2c^2 - cr - \sqrt{4c^3r - 4c^4r + c^2r^2}}{2c^2}\right| < 1$$

otherwise unstable. To verify these criteria for stability, numerical values of parameters c and r must be assigned.

Considering Allee effect on Prey population, the system (2.4) changes into the form

$$x_{n+1} = x_n + rx_n(1 - x_n)\frac{x_n}{u + x_n} - cx_n y_n,$$

$$y_{n+1} = c\, x_n\, y_n \quad (2.6)$$

Here, $\frac{x_n}{u+x_n}$ is, the function of the Allee effect, where u is the Allee constant so that

$$0 < u < \frac{r}{c}.$$

Jacobian matrix of system (2.5), obtained as

$$J = \begin{pmatrix} 1 + \frac{rx(1-x)}{u+x} - \frac{rx^2}{u+x} - \frac{rx^2(1-x)}{(u+x)^2} - cy & -cx \\ cy & cx \end{pmatrix} \quad (2.7)$$

Fixed points of system (2.6) with Allee effect obtained as

$$P_1^{**} = (0,0), \quad P_2^{**} = (1,0) \text{ and } P_3^{**} = \left(\frac{1}{c}, \frac{(c-1)rx}{c(1+cu)}\right).$$

Repeating steps of earlier case for stability, one obtains:

(i) Eigenvalues corresponding to the fixed point $P_1^{**} = (0,0)$, obtained as

$\lambda_1 = 1, \quad \lambda_2 = 0$

which implies that the fixed point $P_1^{**} = (0,0)$ is unstable.

(ii) Eigenvalues corresponding to the fixed point $P_2^{**} = (1,0)$ obtained as

$\lambda_1 = c, \quad \lambda_2 = \frac{1-r+u}{1+u}$

Using conditions of stability, we find that the fixed point $P_2^{**} = (1,0)$ be stable if $0 < c < 1$ and $r > 2(1+u)$.

(iii) Eigenvalues λ_1, λ_2 corresponding to the fixed point $P_3^{**} = \left(\frac{1}{c}, \frac{(c-1)rx}{c(1+cu)}\right)$, appeared with complicated expressions involving parameters c, r and u. So, for simplicity, numerical values of these parameters must be assigned to know the actual criteria of stability of this fixed point.

For example, if we fix values $r = 2.5, c = 2.45$ and $u = 0.03$, then fixed points obtained are:

$P_1^{**} = (0,0)$, $P_2^{**} = (1,0)$ and $P_3^{**} = (0.408163,\ 562566)$.

Then, by stability analysis, we obtain absolute values of eigenvalues

$$|\lambda_1| = 1.23374 \text{ and } |\lambda_2| = 1.23374$$

This implies P_3^{**} is unstable fixed point.

3. Bifurcation Analysis

Bifurcation scenario of the prey-predator system obtained along prey x_n and predator y_n directions by fixing $r = 2.5$ and varying the parameter c for the cases of without application of Allee effect, $u = 0$, and with

application of Allee effect on prey population $u = 0.03$ and $u = 0.1$, as shown in Figure 1. In Figure 2, we have drawn bifurcation happening nearby the periodic windows for all these cases. Looking carefully and comparing corresponding figures in Figure 1 and Figure 2, we can draw conclusion on applications of Allee effects. Within periodic windows, Figure 2, one observes very significant changes due to Allee effect. These magnified these periodic windows, display very amazing scenario of multiperiodic orbits and in some cases, these orbits again evolving into period-doubling and chaos adding phenomena. According the complexity literature, [7, 8, 13, 14, 19, 21, 23 - 26], we observed the phenomena like bi-stability, intermittency etc. visible within these periodic windows. These complex pattern within periodic window are because of the system's inherent complexity. Such complexities may show significant increase and fluctuations in topological entropy for the system. This would be verified after calculating and plotting of topological entropies.

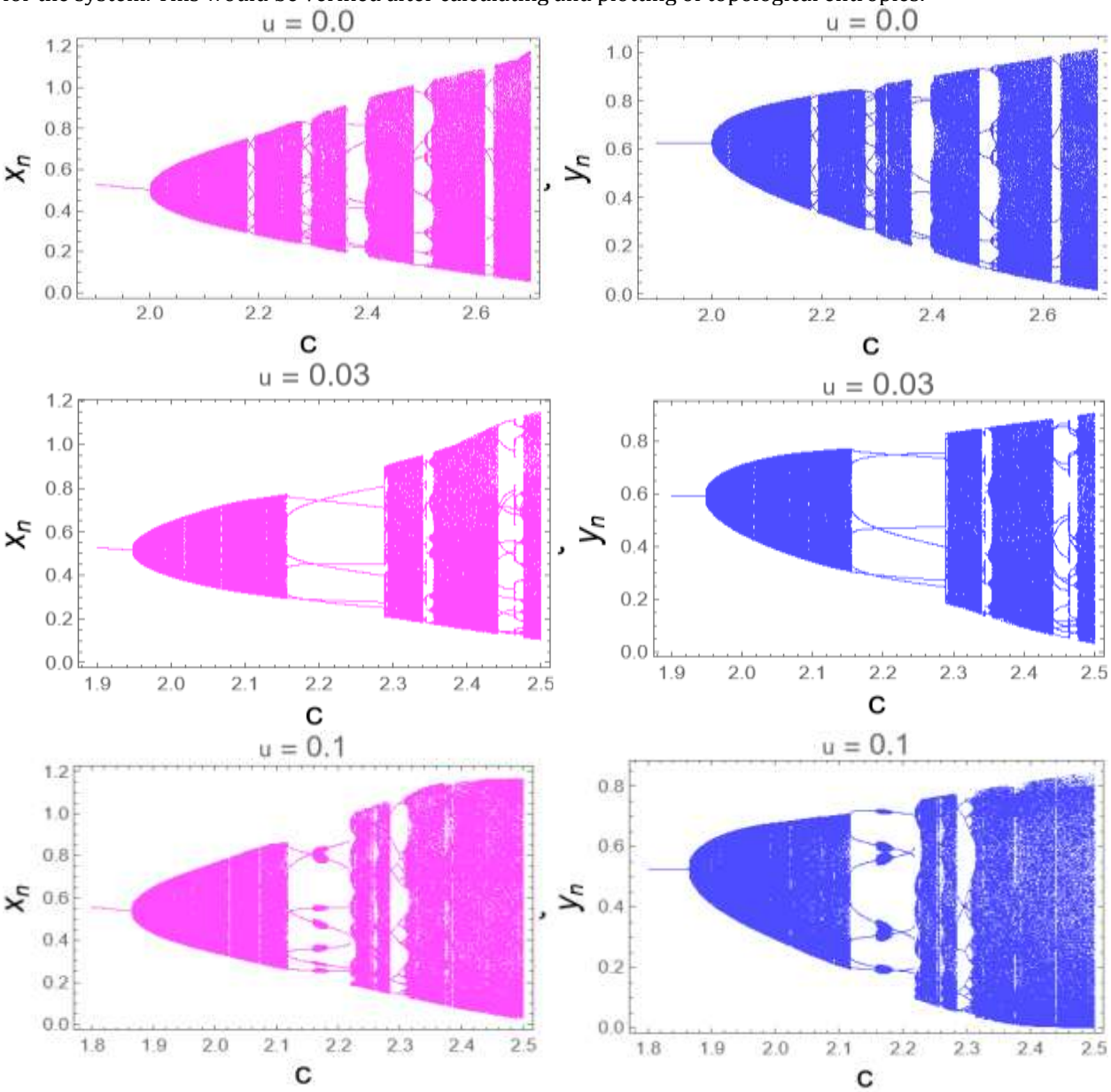


Figure 1: Bifurcation scenario of predator-prey system (5) along prey and predator axes for value of $r = 2.5$ and for cases: (i) $u = 0, 1.9 \leq c \leq 2.7$, (ii) $u = 0.03, 1.9 \leq c \leq 2.7$ and (iii) $u = 0, 1.8 \leq c \leq 2.5$. Lower figures are magnifications of corresponding upper figures within range $2.1 \leq c \leq 2.5$.

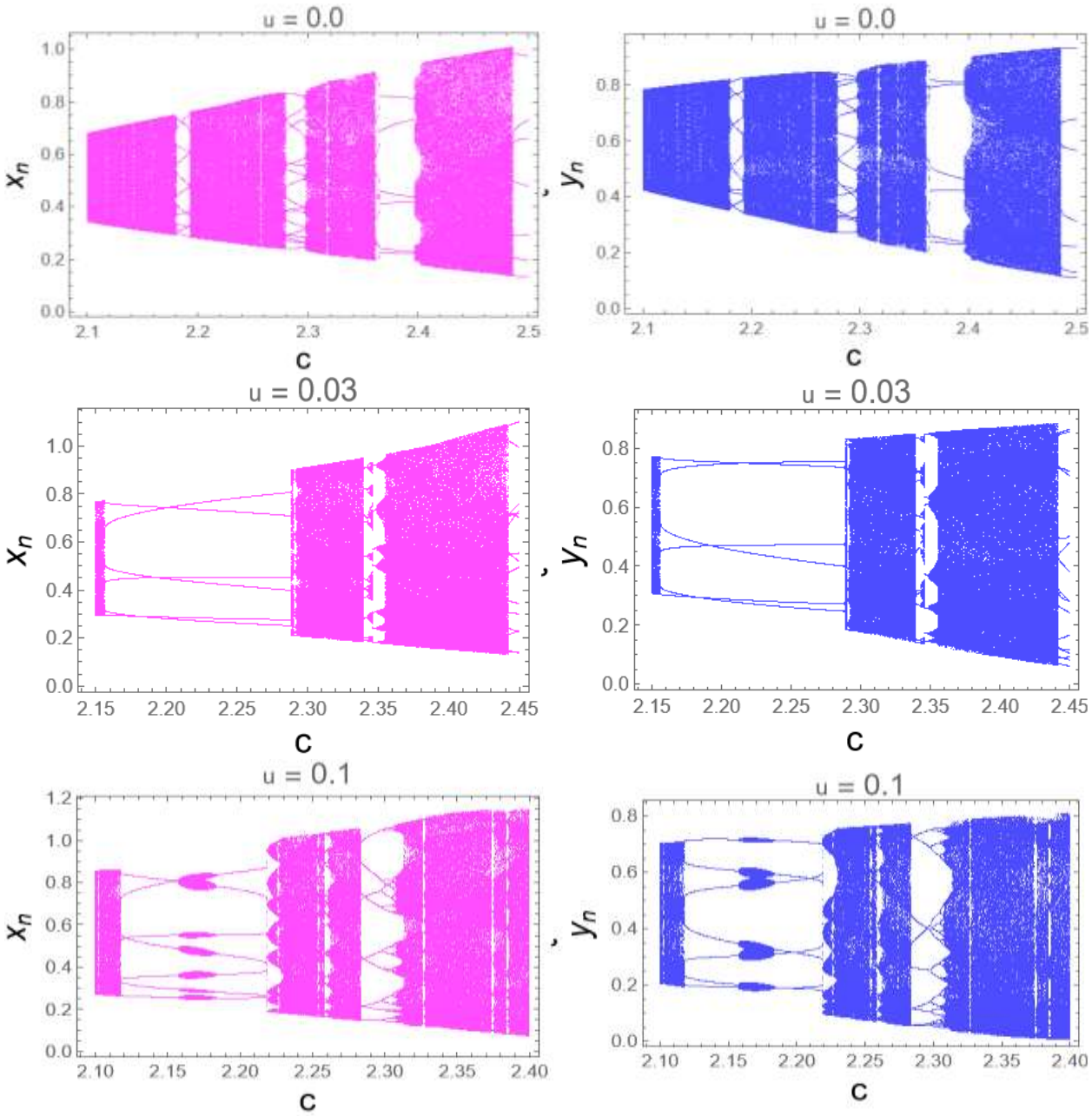


Figure 2: Bifurcation scenario nearby the periodic windows of Figure 1: here, (i) for case $u = 0$, $2.1 \leq c \leq 2.5$, (ii) for case $u = 0.03$, $2.15 \leq c \leq 2.4$ and (iii) for case $u = 0.1$, $2.1 \leq c \leq 2.4$.

4. Results of Numerical Simulations:

(a) **Regular and Chaotic Attractors and Lyapunov Exponents:** Attractors and corresponding Lyapunov exponents are calculated and drawn and presented through Figure 3–Figure 5 for cases without the Allee effect and with the number of preys being affected by the Allee effect. After the Allee effect is applied, the alterations are evident.

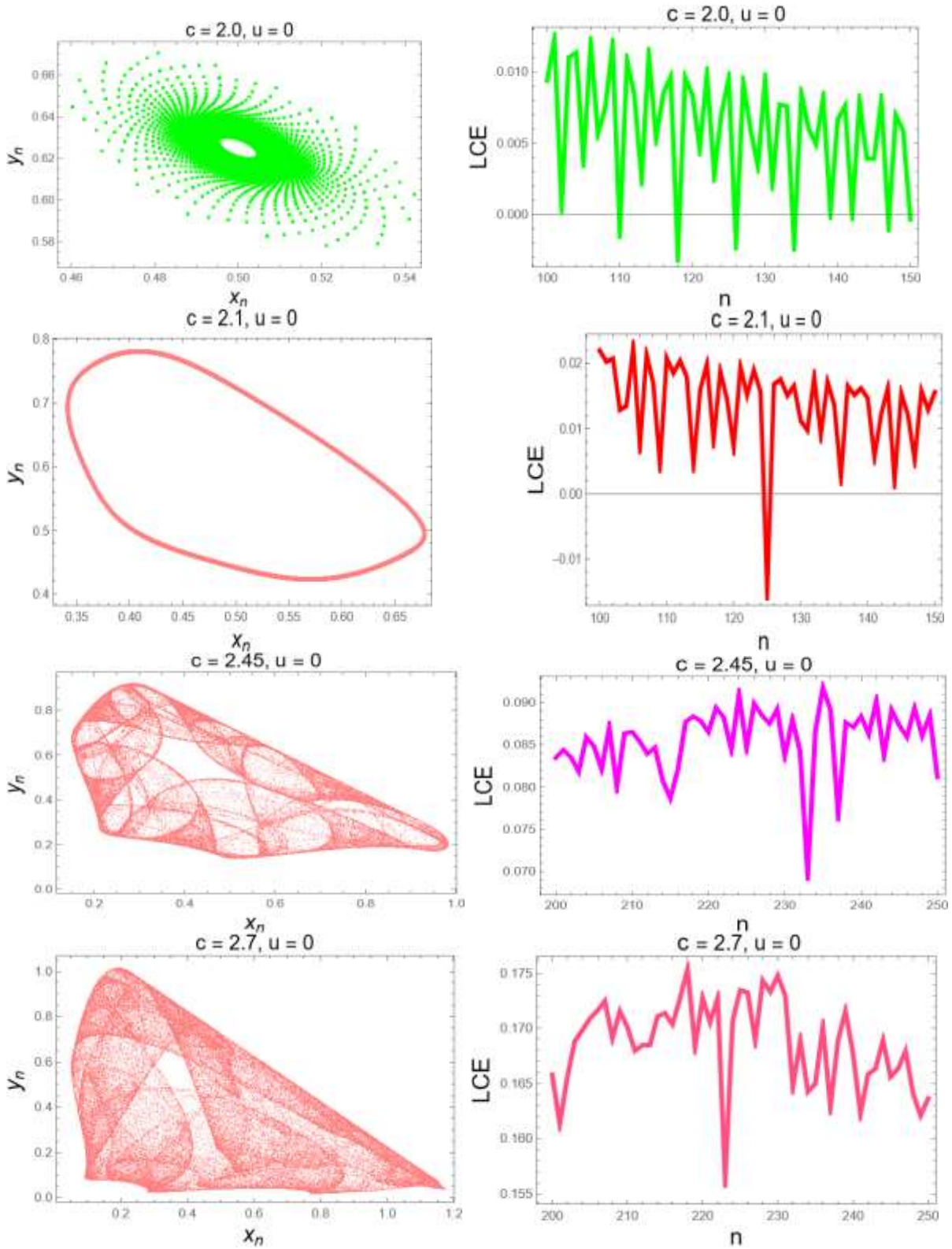


Figure 3: Plots of attractors (left figures) and corresponding LCEs (right figures) for r = 2.5 and different values of parameter c when the Allee effect is not applied to the prey population.

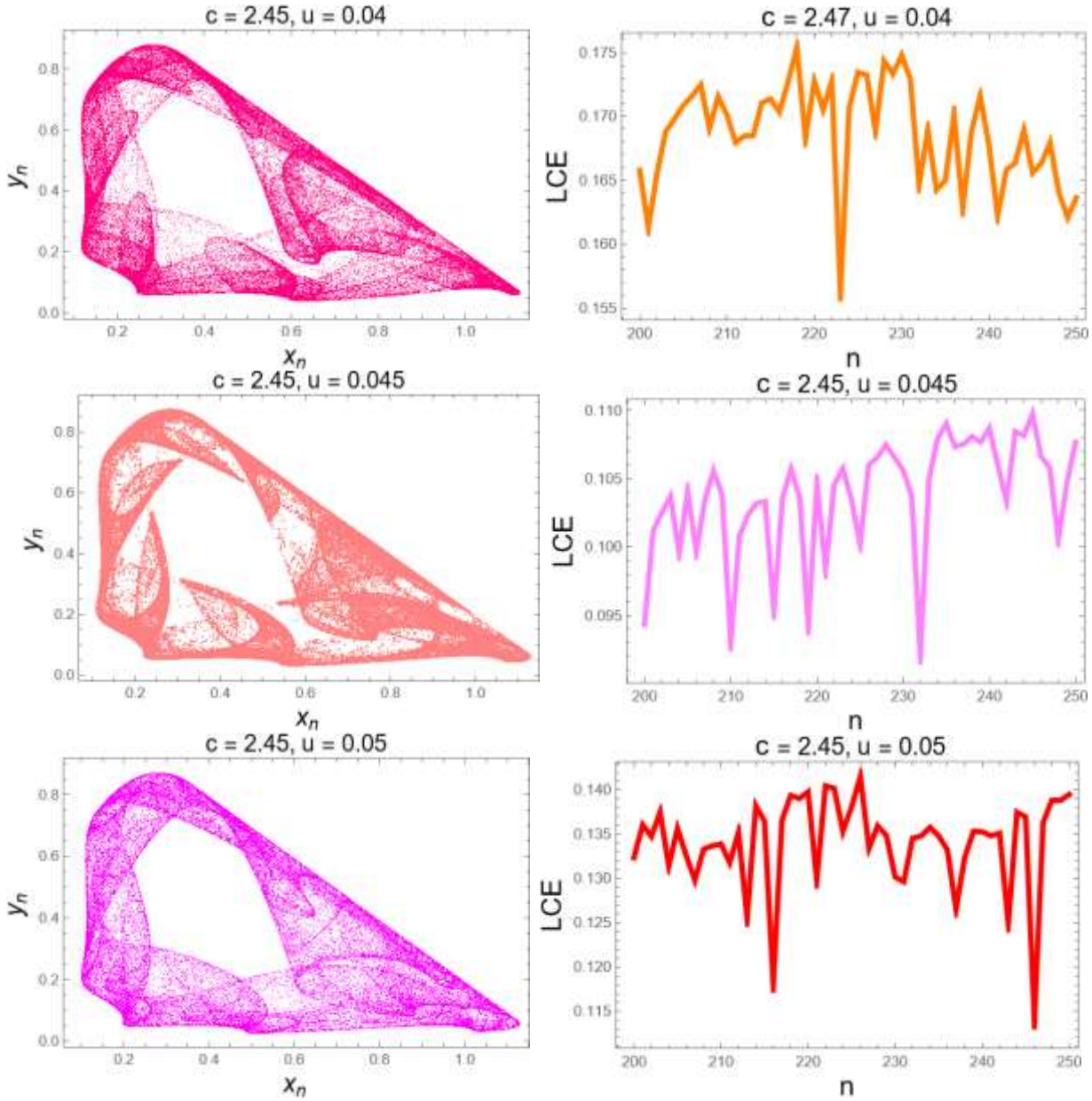
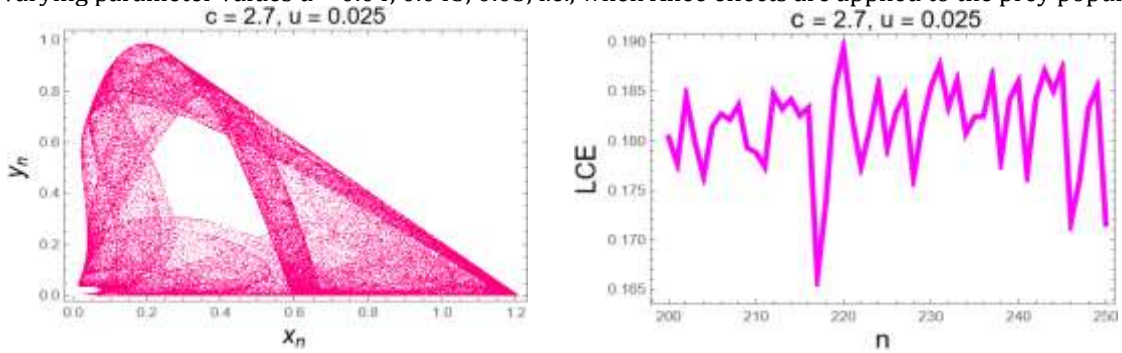


Figure 4: Plots of attractors (left figures) and corresponding LCEs (right figures) for r = 2.5 and c = 2.45, varying parameter values u = 0.04, 0.045, 0.05, i.e., when Allee effects are applied to the prey population.



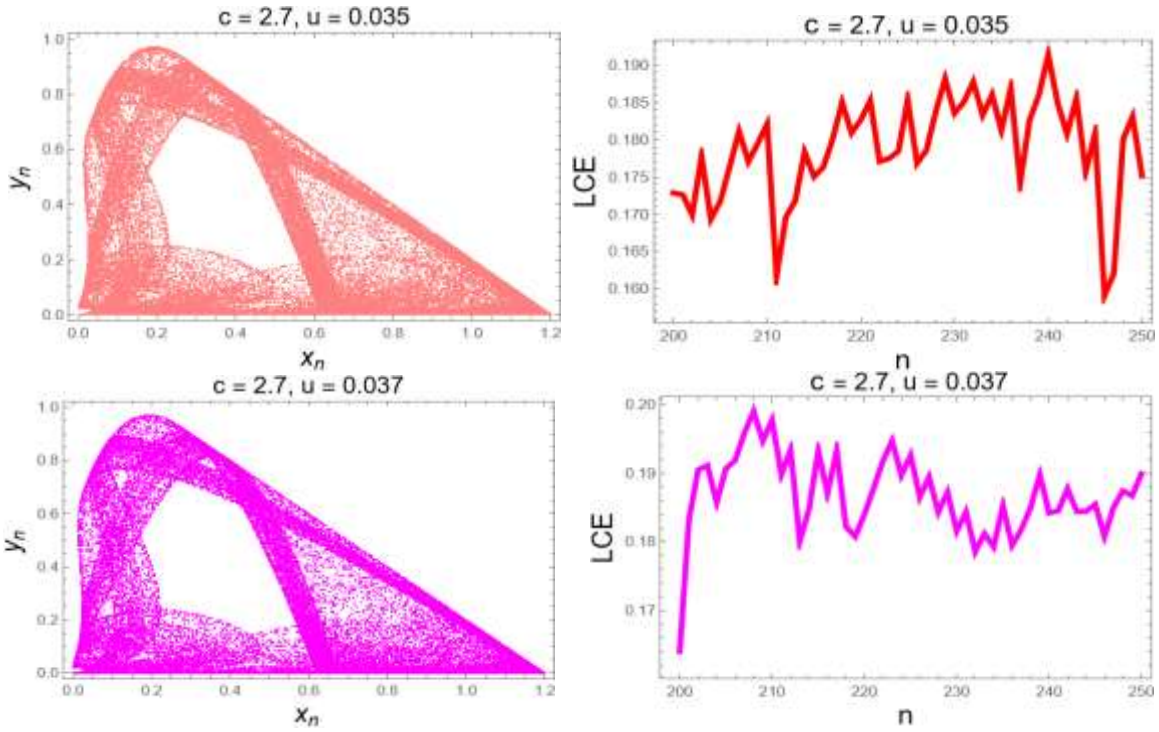


Figure 5: Plots of attractors (left figures) and corresponding LCEs (right figures) for r = 2.5 and c = 2.7, different values of parameter u = 0.025, 0.035, 0.37, i.e., when Allee effects are applied to the prey population.

(***b***) **Calculations of Topological Entropies:**

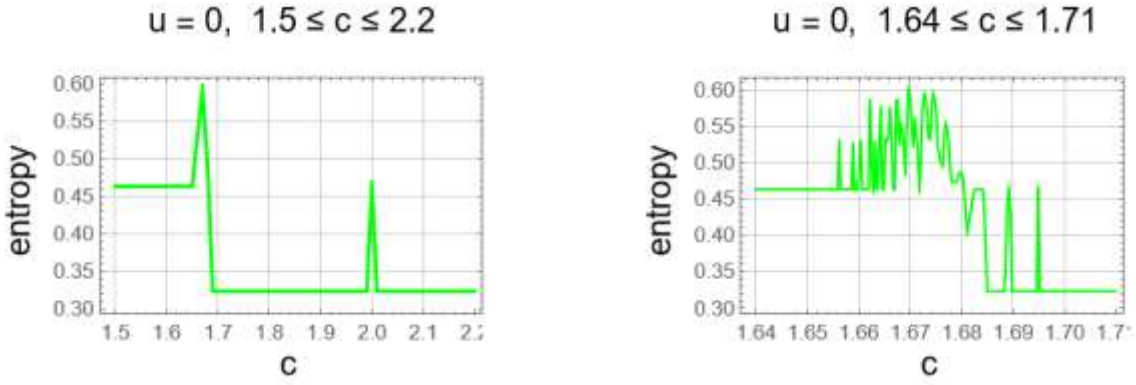
We have the complexity of complex systems measured by the increase and fluctuations of the system's topological entropy. The measure of complexity involves Probability law; it can be used to characterize the rate at which evolutions combine. A dynamical system's rate of mixing is measured via the idea of topological entropy [7, 13, 14, 34–41]. The exponential increase in the number of identifiable orbits over time is measured by topological entropy. Topological entropy is also called Kolmogorov–Sinai entropy [41]. Calculations of topological entropy are as follows:

Examine a state space X that has a finite partition represented by $P = \{A_1, A_2, A_3, \ldots, A_N\}$. The probability of a given reading is thus defined as a measure μ on X with total measure μ(X) = 1.

$$p_i = \mu(A_i), i = 1, 2, .., N. \quad (2.8)$$

After then, the partition's entropy is determined by

$$H(P) = -\sum_{i=0}^{N} p_i \, Log \, p_i \quad (2.9)$$



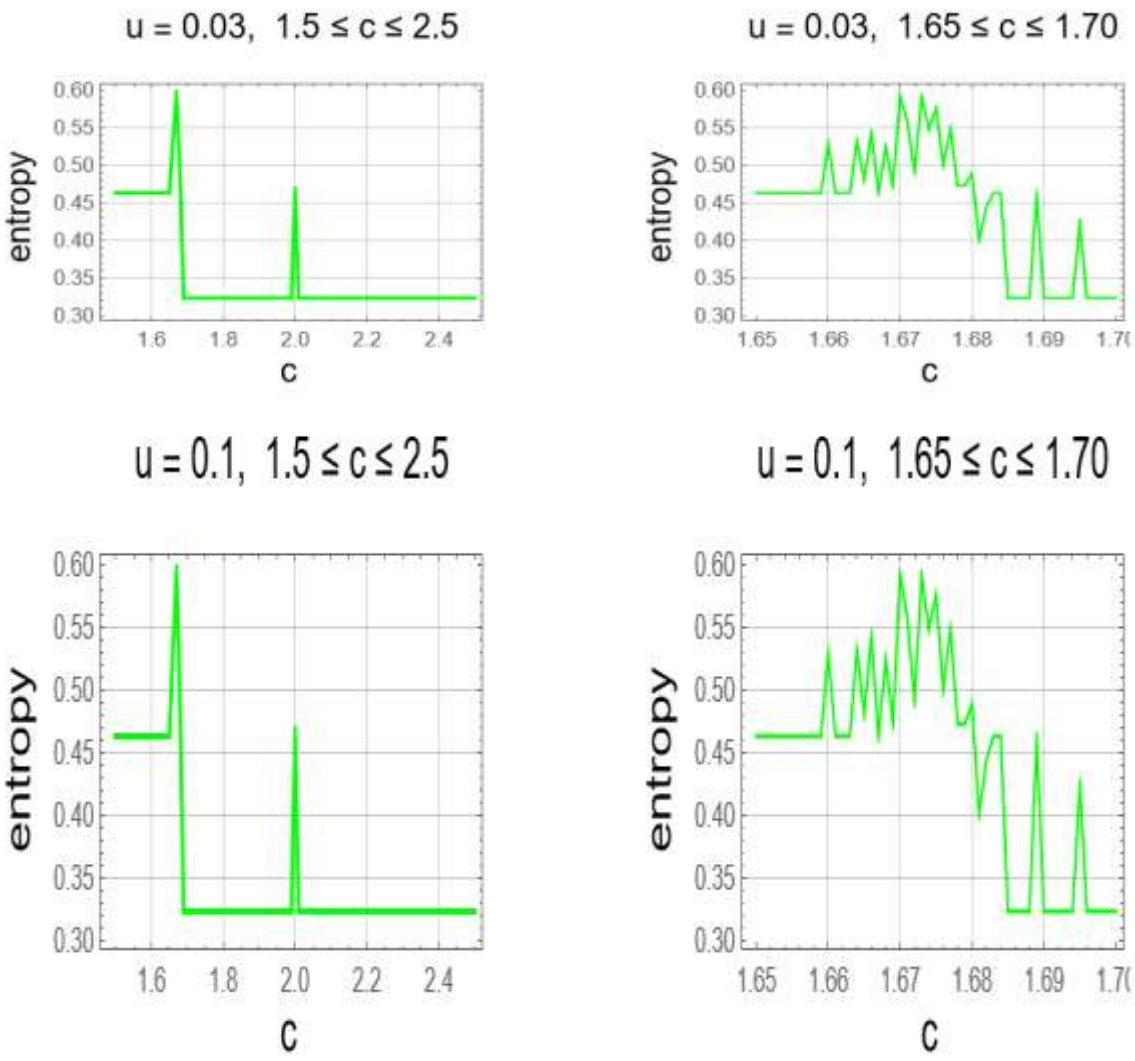


Figure 6: Plots of topological entropies when r = 2.5 and different values, u = 0 and u = 0.1 of the Allee constant. Right column figures represent extended plots nearby the pick.

(c): Correlation Dimensions:

Correlation dimension, the chaotic attractor's dimensionality is measured by a kind of fractal dimension. Every chaotic attraction, (sometimes called as strange attractor), possess property such as self-similarity and non-integer dimensions. Correlation dimension of any chaotic attractor calculated done statistically with the Heaviside function applied, [50].

First, for a certain $r \ll 1$, we compute the correlation integral data C(r). The correlation data curve is then plotted, by plotting $\frac{log\, C(r)}{log\, r}$ against r, as depicted in Figure 8. Next, use a criterion for linear fit, (linear regression) to determine the equation of the straight line that almost fits the data points and to compare it to the correlation data. The correlation dimension is provided by this line's y-intercept. D_C .

Following above steps, correlation dimension for chaotic attractor obtained for $r = 2.5$, $c = 2.7$ and $u = 0.025$, obtained as follows:

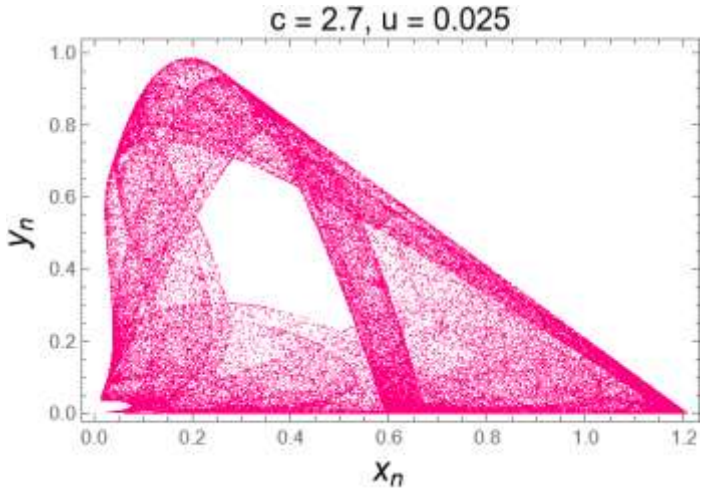


Figure 7: Plot of chaotic attractor for $r = 2.5$, $c = 2.7$ and $u = 0.025$.

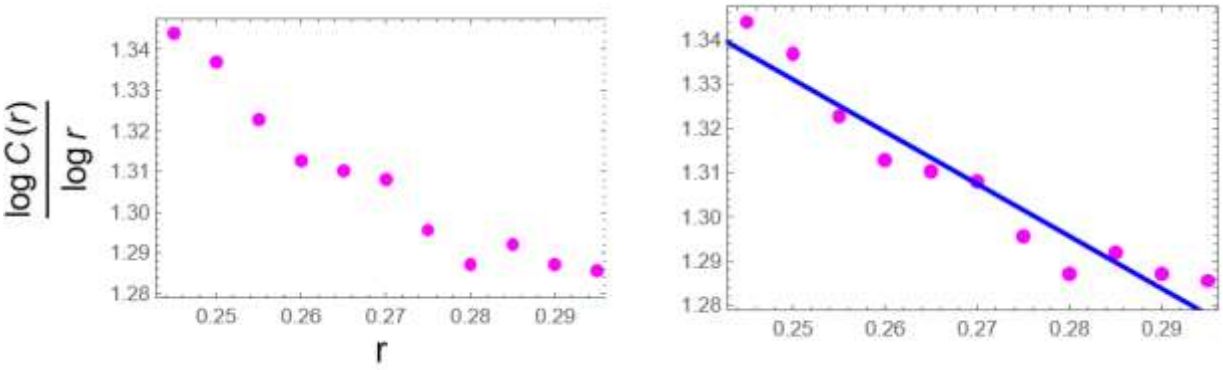


Figure 8: Left plot represents the correlation curve and the right plot represents linear fitting of the data points of the correlation curve.

The equation for the straight line that roughly fits the correlation data point curve's points is as follows:

$$y = 1.626 - 1.17988x \quad (2.10)$$

Thus, the chaotic attractor's correlation dimension, which was acquired for $r = 2.5$, $u = 0.254$ and $c = 2.7$ is obtained as $D_c = 1.626 \cong 1.63$. Repeating the procedure, correlation dimensions of some chaotic attractors obtained for this system shown in Table 1.

Table 1: Calculations of Correlation Dimensions D_c of Chaotic Attractors:

r	u	c	D_c
$r = 2.5$	0.0	2.45	1.53
	0.0	2.7	1.79
	0.04	2.45	1.36
	0.045	2.45	1.33
	0.05	2.45	1.56
	0.025	2.7	1.47
	0.035	2.7	0.954
	0.037	2.7	1.044

5. Concluding Remarks:

A detailed study was performed on a system of prey and predators that is susceptible to the Allee effect on the prey population. Different bifurcation scenarios of the system (Figure 1 & Figure 2) for cases of null Allee effect, u = 0, and with Allee effect, u = 0, 0.03, 0.1 on prey population, clearly display the consequences of the application of the Allee effect. Bifurcations near the periodic windows appeared to have complex patterns for all the cases but with significant changes when the Allee effect is in action. Periodic windows within chaotic

regions are displaying amazing scenarios of multiperiodic orbits, and in some cases, these orbits are again evolving into period-doubling and chaos, adding phenomena. The phenomena like bi-stability, intermittency, etc., can also be seen within these periodic windows, which are in confirmation that the system under observation is a complex system.

Attractors of the evolving prey-predator system are together with Lyapunov exponents obtained and shown through figures, Figure 3–Figure 5. One can mark here the differences between attractors and their corresponding plots of Lyapunov exponents for similar cases with and without the application of the Allee effect on prey populations.

As a complexity measure, plots of topological entropies are obtained for three cases: u = 0, 0.03, and 0.1, Figure 6. One observes a very significant amount of increase in topological entropy, which is nearly the same for all cases, justifying the presence of complexity. Thus, the Allee effect on prey growth does not reduce the dynamical complexity of the system, as claimed in the recent article [49]. Little differences in fluctuations of topological entropies appeared, right column figures, Figure 6, but not very significant.

Correlation dimensions, a type of fractal dimension, of chaotic attractors calculated for cases without Allee effect, (u = 0) and with application of Allee effect on prey population obtained and presented in Table 1. Steps of procedure of calculations explained in detail. Looking at these correlation dimensions, definitely, one finds change occurring with application of Allee effect.

The findings of this study are important and hope, these results will contribute significantly in Mathematical Biological problem. For numerical simulation works in this study we have used Mathematica codes described in the books by Martelli, [50}, and Lynch, [51].

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